MODIFIED FACTOR-TYPE ESTIMATORS WITH TWO AUXILIARY VARIABLES UNDER TWO-PHASE SAMPLING

A. Audu [1], A. A. Adewara [2]

[1] **Department of Mathematics, Usmanu Danfodiyo University, P.M.B. 2346, Sokoto, Nigeria**
[2] **Department of Statistics, University of Ilorin, P.M.B 1515, Ilorin, Kwara State, Nigeria**

Corresponding author: A. Audu, ahmed.audu@udusok.edu.ng

ABSTRACT: In this paper, two modified factor-type estimators with two auxiliary variables for population mean have been suggested. Bias and MSE of the suggested estimators have been derived up to first order approximation using Tailor's series expansion and the conditions for their efficiency over some existing estimators have been established theoretically. Empirical study was conducted using three dataset and the results revealed that the suggested estimators are more efficient.
KEYWORDS: Estimator, Auxiliary variable, Mean square error (MSE), Two-phase sampling.

1.1 INTRODUCTION

Use of known functions of auxiliary variables in improving the performance of estimators has been one of the major strategies that have received wide attention of several authors like Chand ([Cha75]), Singh ([Sin01]), Singh and Upadhayaya ([SU01]), Singh et al. ([SUC04, SCS07]), Khan et al. ([KSS12]), Choudhury and Singh ([CS12]) in sampling survey. According to Choudhury and Singh ([CS12]), if auxiliary variable X is strongly correlated with another variable Z, the known information on Z like coefficients of kurtosis, skewness and variation e.t.c. help in increasing the efficiency of estimators if they are judiciously utilized.
Let $\Omega = (1\ 2, 3 \ldots N)$be a population of size N and Y, X, Z be three real valued functions having values$(Y_i,\ X_i, Z_i) \in \mathbb{R}^+ > 0$ on the i^{th} unit of $U(1 \le i \le N)$. Let $\bar{Y}$, $\bar{X}$ and $\bar{Z}$ be the population means of Y, X and Z respectively with C_y, C_x and $C_z$as coefficients of variation of Y, X and Z respectively.

Notations: $A = (d-1)(d-2)$, $B = (d-1)(d-4)$, $C = (d-2)(d-3)(d-4)$,

d is an unknown positive real number to be estimated i.e $d \in \Re^+$

$$\psi_1 = \frac{A+C}{A+fB+C}, \psi_2 = \frac{fB}{A+fB+C}, \psi_3 = \frac{A+fB}{A+fB+C}, \psi_4 = \frac{C}{A+fB+C} \quad , P = \psi_3 - \psi_1 = \psi_2 - \psi_4$$

Shukla ([Shu02]) suggested a factor-type estimator for population mean under two-phase sampling as

$$\bar{y}_{FTd} = \bar{y}\frac{(A+C)\bar{x}_1 + fB\bar{x}}{(A+fB)\bar{x}_1 + C\bar{x}} \tag{1.1}$$

The bias and MSE of $\bar{y}_{FTd}$ under case I and II are respectively

$$Bias(\bar{y}_{FTd})_I = \bar{Y}P\theta_3\left[\rho_{xy}C_xC_y - \psi_4 C_x^2\right] \tag{1.2}$$

$$Bias(\bar{y}_{FTd})_{II} = \bar{Y}P\left[(\theta_1\psi_3 - \theta_2\psi_4)C_x^2 + \theta_2\rho_{xy}C_xC_y\right] \tag{1.3}$$

$$MSE(\bar{y}_{FTd})_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P^2 C_x^2 + 2\theta_3 P\rho_{xy}C_xC_y\right] \tag{1.4}$$

$$MSE(\bar{y}_{FTd})_{II} = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_4 P^2 C_x^2 + 2\theta_2 P\rho_{xy}C_xC_y\right] \tag{1.5}$$

where $\theta_1 = \frac{1}{n_1} - \frac{1}{N}$, $\theta_2 = \frac{1}{n_2} - \frac{1}{N}$, $\theta_3 = \frac{1}{n_2} - \frac{1}{n_1}$, $\theta_4 = \theta_1 + \theta_2$

2.0 SUGGESTED ESTIMATORS

Motivated by the work of Choudhury and Singh ([CS12]), the following factor-type estimators are suggested

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{y}_2 \left[\frac{(A+C)\bar{x}_1 + fB\bar{x}_2}{(A+fB)\bar{x}_1 + C\bar{x}_2} \right] \left[\frac{a_z\bar{Z} + b_z}{a_z\bar{z}_1 + b_z} \right]^{\beta_1} \tag{2.1}$$

$$\bar{y}_{FTAA}^{\beta_2(d)} = \bar{y}_2 \left[\frac{(A+C)\bar{x}_1 + fB\bar{x}_2}{(A+fB)\bar{x}_1 + C\bar{x}_2} \right] \left[\frac{a_z\bar{z}_1 + b_z}{a_z\bar{z}_2 + b_z} \right]^{\beta_2} \tag{2.2}$$

Where $0 < \beta_1 < 1,\ 0 < \beta_2 < 1$

3.0 PROPERTIES OF THE PROPOSED FACTOR-TYPE ESTIMATORS

In this section, the theoretical biases and MSEs of the suggested estimators are derived up to first order approximation using Taylor's series expansion.
In order to study the properties of the proposed estimators, we define the following error terms $\epsilon_{\bar{y}_2} = (\bar{y}_2 - \bar{Y})/\bar{Y}$, $\epsilon_{\bar{x}_1} = (\bar{x}_1 - \bar{X})/\bar{X}$, $\epsilon_{\bar{x}_2} = (\bar{x}_2 - \bar{X})/\bar{X}$, $\epsilon_{\bar{z}_1} = (\bar{z}_1 - \bar{Z})/\bar{Z}$, $\epsilon_{\bar{z}_2} = (\bar{z}_2 - \bar{Z})/\bar{Z}$, such that $|\epsilon_{\bar{y}_2}| < 1$, $|\epsilon_{\bar{x}_1}| < 1$, $|\epsilon_{\bar{x}_2}| < 1$, $|\epsilon_{\bar{z}_1}| < 1$, $|\epsilon_{\bar{z}_2}| < 1$

Under case I: When secondary sample is a subset of preliminary sample $(S_2 \subset S_1)$

$$\left.\begin{array}{l}
\bar{y}_1 = \bar{Y}(1+\epsilon_{\bar{y}_1}), \quad \bar{x}_1 = \bar{X}(1+\epsilon_{\bar{x}_1}), \quad \bar{z}_1 = \bar{Z}(1+\epsilon_{\bar{z}_1}) \\
\bar{y}_2 = \bar{Y}(1+\epsilon_{\bar{y}_2}), \quad \bar{x}_2 = \bar{X}(1+\epsilon_{\bar{x}_2}), \quad \bar{z}_2 = \bar{Z}(1+\epsilon_{\bar{z}_2}) \\
E(\epsilon_{\bar{x}_1}) = E(\epsilon_{\bar{z}_1}) = E(\epsilon_{\bar{y}_2}) = E(\epsilon_{\bar{x}_2}) = E(\epsilon_{\bar{z}_2}) = 0 \\
E(\epsilon_{\bar{y}_2}^2) = \theta_2 C_y^2, \quad E(\epsilon_{\bar{x}_1}^2) = \theta_1 C_x^2, \quad E(\epsilon_{\bar{z}_1}^2) = \theta_1 C_z^2, \; E(\epsilon_{\bar{x}_2}^2) = \theta_2 C_x^2 \\
E(\epsilon_{\bar{z}_2}^2) = \theta_2 C_z^2, \quad E(\epsilon_{\bar{y}_2}\epsilon_{\bar{x}_1}) = \theta_1 \rho_{xy} C_y C_x, \; E(\epsilon_{\bar{y}_2}\epsilon_{\bar{z}_1}) = \theta_1 \rho_{yz} C_y C_z \\
E(\epsilon_{\bar{y}_2}\epsilon_{\bar{x}_2}) = \theta_2 \rho_{xy} C_y C_x, \; E(\epsilon_{\bar{y}_2}\epsilon_{\bar{z}_2}) = \theta_2 \rho_{yz} C_y C_z, \; E(\epsilon_{\bar{x}_1}\epsilon_{\bar{z}_1}) = \theta_1 \rho_{xz} C_x C_z \\
E(\epsilon_{\bar{x}_1}\epsilon_{\bar{z}_2}) = \theta_1 \rho_{xz} C_x C_z, \; E(\epsilon_{\bar{x}_1}\epsilon_{\bar{x}_2}) = \theta_1 C_x^2, \quad E(\epsilon_{\bar{x}_2}\epsilon_{\bar{z}_1}) = \theta_1 \rho_{xz} C_x C_z \\
E(\epsilon_{\bar{x}_2}\epsilon_{\bar{z}_2}) = \theta_2 \rho_{xz} C_x C_z, E(\epsilon_{\bar{z}_1}\epsilon_{\bar{z}_2}) = \theta_1 C_z^2, \; \theta_1 = \frac{1}{n_1} - \frac{1}{N}, \; \theta_2 = \frac{1}{n_2} - \frac{1}{N}, \; \theta_3 = \theta_2 - \theta_1
\end{array}\right\} \tag{3.1}$$

Under case II: When secondary sample is a subset of population $(S_2 \subset \Omega_N)$

$$\left.\begin{array}{l}
\bar{y}_1 = \bar{Y}(1+\epsilon_{\bar{y}_1}), \quad \bar{x}_1 = \bar{X}(1+\epsilon_{\bar{x}_1}), \quad \bar{z}_1 = \bar{Z}(1+\epsilon_{\bar{z}_1}) \\
\bar{y}_2 = \bar{Y}(1+\epsilon_{\bar{y}_2}), \quad \bar{x}_2 = \bar{X}(1+\epsilon_{\bar{x}_2}), \quad \bar{z}_2 = \bar{Z}(1+\epsilon_{\bar{z}_2}) \\
E(\epsilon_{\bar{x}_1}) = E(\epsilon_{\bar{z}_1}) = E(\epsilon_{\bar{y}_2}) = E(\epsilon_{\bar{x}_2}) = E(\epsilon_{\bar{z}_2}) = 0 \\
E(\epsilon_{\bar{y}_2}^2) = \theta_2 C_y^2, \quad E(\epsilon_{\bar{x}_1}^2) = \theta_1 C_x^2, \quad E(\epsilon_{\bar{z}_1}^2) = \theta_1 C_z^2, \; E(\epsilon_{\bar{x}_2}^2) = \delta_2 C_x^2 \\
E(\epsilon_{\bar{z}_2}^2) = \theta_2 C_z^2, E(\epsilon_{\bar{y}_2}\epsilon_{\bar{x}_2}) = \theta_2 \rho_{xy} C_y C_x, \; E(\epsilon_{\bar{y}_2}\epsilon_{\bar{z}_2}) = \theta_2 \rho_{yz} C_y C_z, \\
E(\epsilon_{\bar{x}_1}\epsilon_{\bar{z}_1}) = \theta_1 \rho_{xz} C_x C_z, \; E(\epsilon_{\bar{x}_2}\epsilon_{\bar{z}_2}) = \theta_2 \rho_{xz} C_x C_z \\
E(\epsilon_{\bar{x}_1}\epsilon_{\bar{z}_2}) = E(\epsilon_{\bar{x}_1}\epsilon_{\bar{x}_2}) = E(\epsilon_{\bar{x}_2}\epsilon_{\bar{z}_1}) = E(\epsilon_{\bar{y}_2}\epsilon_{\bar{x}_1}) = E(\epsilon_{\bar{y}_2}\epsilon_{\bar{z}_1}) = E(\epsilon_{\bar{z}_1}\epsilon_{\bar{z}_2}) = 0
\end{array}\right\} \tag{3.2}$$

3.1 Bias and MSE of the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$

The suggested factor-type estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ can be expressed in terms of error terms $\in_{\bar{x}_1}, \in_{\bar{x}_2}, \in_{\bar{z}_1}$ and $\in_{\bar{y}_2}$ is

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{Y}\left[1+\in_{\bar{y}_2}\right]\left[1+\psi_1 \in_{\bar{x}_1} + \psi_2 \in_{\bar{x}_2}\right]\left[1+\psi_3 \in_{\bar{x}_1} + \psi_4 \in_{\bar{x}_2}\right]^{-1}\left[1+\delta_z \in_{\bar{z}_1}\right]^{-\beta_1} \tag{3.3}$$

Here the assumption is that in (3.3), $\left|\psi_3 \in_{\bar{x}_1} + \psi_4 \in_{\bar{x}_2}\right| < 1$ and $\left|\delta_z \in_{\bar{z}_1}\right| < 1$ so that $\left(1+\psi_3 \in_{\bar{x}_1} + \psi_4 \in_{\bar{x}_2}\right)^{-1}$ and $\left(1+\delta_z \in_{\bar{z}_1}\right)^{-\beta_1}$ are expandable.

Subtract $\bar{Y}$ for both side of (3.3) and using power series expansion, the simplification of (3.3) up to first order approximation $O(n^{-1})$ is given by

$$\begin{aligned}\bar{y}_{FTAA}^{\beta_1(d)} - \bar{Y} = \bar{Y}\Bigg[&\in_{\bar{y}_2} - \beta_1\delta_z \in_{\bar{z}_1} - P\in_{\bar{x}_1} + P\in_{\bar{x}_2} + \frac{\beta_1(\beta_1+1)}{2}\delta_z^2 \in_{\bar{z}_1}^2 + \beta_1\delta_z P \in_{\bar{x}_1}\in_{\bar{z}_1} - \beta_1\delta_z \in_{\bar{x}_2}\in_{\bar{z}_1} + \\ &\psi_3 P \in_{\bar{x}_1}^2 - \psi_4 P \in_{\bar{x}_2}^2 + \left(2\psi_3\psi_4 - \psi_1\psi_4 - \psi_2\psi_3\right)\in_{\bar{x}_1}\in_{\bar{x}_2} - \beta_1\delta_z \in_{\bar{y}_2}\in_{\bar{z}_1} - P\in_{\bar{y}_2}\in_{\bar{x}_1} + P\in_{\bar{y}_2}\in_{\bar{x}_2}\Bigg]\end{aligned} \tag{3.4}$$

Taking expectation of (3.4) and apply the results in (3.1), the bias of the suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I is obtained as

$$Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}\left[\theta_3 P C_x^2\left(\rho_{xy}C_xC_y - \psi_4\right) + \theta_1 C_z^2\left(\frac{\beta_1(\beta_1+1)}{2}\delta_z^2 - \beta_1\delta_z\rho_{yz}C_yC_z\right)\right] \tag{3.5}$$

Also, taking expectation of (3.4) and using results of (3.2), the bias of the suggested estimator to terms of order n^{-1} under case II is obtained as

$$Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}\left[\theta_3 P\psi_3 C_x^2 + P\theta_2 C_x^2\left(C_{yx} - \psi_4\right) + \beta_1\delta_z\theta_3 C_z^2\left(\delta_z\left(\beta_1+1\right)/2 + PC_{xz}\right)\right] \tag{3.6}$$

Square both sides of (3.4), taking expectation and using the results in (3.1), we obtain the MSE of the suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I as:

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2 P\left(P + 2C_{yx}\right) + \theta_1\beta_1\delta_z C_z^2\left(\beta_1\delta_z - 2C_{yz}\right)\right] \tag{3.7}$$

Also, square both sides of (3.4), taking expectation and using the results in (3.2), we obtain the MSE of the suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case II as:

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}^2\left[\theta_2 C_y^2 + C_x^2 P\left\{\theta_2\left(P + 2C_{yx}\right) + \theta_3 P\right\} + \theta_3\beta_1\delta_z C_z^2\left(\beta_1\delta_z + 2PC_{xz}\right)\right] \tag{3.8}$$

Differentiate equation (3.7) partially with respect to P and equate to zero,

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2\left[2P\theta_3 C_x^2 + 2\theta_3\rho_{xy}C_xC_y\right] = 0 \tag{3.9}$$

$$P = -C_{yx} = P_{31}^{opt} \tag{3.10}$$

Differentiate equation (3.7) partially with respect to β_1 and equate to zero,

$$\frac{\partial}{\partial \beta_1} MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2\left[2\theta_1 C_z^2\left(\beta_1\delta_z^2 - \delta_z C_{xz}\right)\right] = 0 \tag{3.11}$$

$$\beta_1 = C_{yz} / \delta_z = \beta_{11}^{opt} \tag{3.12}$$

Substituting the values of P_{31}^{opt} and β_{11}^{opt} in equation (3.7) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I as

$$MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2 C_y^2\left[\theta_2 - \theta_3\rho_{xy}^2 - \theta_1\rho_{yz}^2\right] \tag{3.13}$$

In order to estimate unknown constant $d \in \Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I, $P = -C_{yx}$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = -C_{yx} \tag{3.14}$$

$$\frac{fB - C}{A + C + fB} = -C_{yx} \tag{3.15}$$

$$\begin{aligned}(u-1)d^3 + (fu + f - 8u + 9)d^2 - (5fu + 5f - 23u + 26)d \\ + (4fu + 4f - 22u + 24) = 0\end{aligned} \tag{3.16}$$

where $u = -C_{yx}$

By solving (3.16), at most 3 zeros d_1, d_2 and d_3 of the polynomials for which (2.1) is optimal under case I will be obtained.

Also, differentiate equation (3.8) with respect to P and β_1 and equate to zero, we obtain:

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}^2\left[2\theta_2 C_x^2\left(P + C_{yx}\right) + 2\theta_1\left(PC_x^2 + \beta_1\delta_z C_z^2 C_{xz}\right)\right] = 0 \tag{3.17}$$

$$P = \frac{-\left[\theta_2 C_x^2 C_{yx} + \theta_1\beta_1\delta_z C_z^2 C_{xz}\right]}{\theta_4 C_x^2} \tag{3.18}$$

$$\frac{\partial}{\partial \beta_1} MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}^2\theta_1\left[2C_z^2\left(\beta_1\delta_z^2 + \delta_z P C_{xz}\right)\right] = 0 \tag{3.19}$$

$$\beta_1 = -PC_{xz} / \delta_z \tag{3.20}$$

Solving P and β_1 simultaneously, the optimum values of P and β_1 are

$$P = -\theta_2 C_{yx} / \left(\theta_4 - \theta_1\rho_{xz}^2\right) = P_{32}^{opt} \tag{3.21}$$

$$\beta_{12}^{opt} = \frac{-\theta_2 C_{xy} C_{xz}}{\delta_z\left(\theta_4 - \theta_1\rho_{xz}^2\right)} \tag{3.22}$$

Substituting the values of P_{32}^{opt} and β_{12}^{opt} in equation (3.8) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_1(d)}$ under case II as

$$MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}^2\theta_2 C_y^2\left[1 - \rho_{xy}^2 / \left(1 + \left(\theta_1/\theta_2\right) - \left(\theta_1/\theta_2\right)\rho_{xz}^2\right)\right] \tag{3.23}$$

In order to estimate unknown constant $d \in \Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case II, $P = -\theta_2 C_{yx} / \left(\theta_4 - \theta_1\rho_{xz}^2\right)$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3-\psi_1=-\theta_2 C_{yx}/\left(\theta_4-\theta_1\rho_{xz}^2\right) \tag{3.24}$$

$$\frac{fB-C}{A+C+fB}=-\theta_2 C_{yx}/\left(\theta_4-\theta_1\rho_{xz}^2\right) \tag{3.25}$$

$$\begin{aligned}(t-1)d^3+(ft+f-8t+9)d^2-(5ft+5f-23t+26)d&\\ +(4ft+4f-22t+24)=0&\end{aligned} \tag{3.26}$$

where $t=-\theta_2 C_{yx}/\left(\theta_4-\theta_1\rho_{xz}^2\right)$

By solving (3.26), at most 3 zeros d_1, d_2 and d_3 of the polynomials for which (2.1) is optimal under case II will be obtained.

3.2 Bias and MSE of the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$

To the first degree of approximation, the suggested factor-type estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ can be expressed in terms of error terms $\in_{\bar{x}_1}, \in_{\bar{x}_2}, \in_{\bar{z}_1}, \in_{\bar{z}_2}$ and $\in_{\bar{y}_2}$ as

$$\bar{y}_{FTAA}^{\beta_2(d)}=\bar{Y}\left[1+\in_{\bar{y}_2}\right]\left[1+\psi_1\in_{\bar{x}_1}+\psi_2\in_{\bar{x}_2}\right]\left[1+\psi_3\in_{\bar{x}_1}+\psi_4\in_{\bar{x}_2}\right]^{-1}\left[1+\delta_z\in_{\bar{z}_1}\right]^{-\beta_2}\left[1+\delta_z\in_{\bar{z}_2}\right]^{-\beta_2} \tag{3.27}$$

Here, we now assume that in (3.92), $\left|\psi_3\in_{\bar{x}_1}+\psi_4\in_{\bar{x}_2}\right|<1$, $\left|\delta_z\in_{\bar{z}_1}\right|<1$ and $\left|\delta_z\in_{\bar{z}_2}\right|<1$ so that $\left(1+\psi_3\in_{\bar{x}_1}+\psi_4\in_{\bar{x}_2}\right)^{-1}$, $\left(1+\delta_z\in_{\bar{z}_1}\right)^{-\beta_2}$ and $\left(1+\delta_z\in_{\bar{z}_2}\right)^{-\beta_2}$ are expandable.

Expanding the right hand side of (3.27) up to second degree approximation and then subtract $\bar{Y}$, we have,

$$\begin{aligned}\bar{y}_{FTAA}^{\beta_2(d)}-\bar{Y}=\bar{Y}\Bigg[&\in_{\bar{y}_2}-\beta_2\delta_z\in_{\bar{z}_2}-P\in_{\bar{x}_1}+P\in_{\bar{x}_2}+\frac{\beta_2(\beta_2+1)}{2}\delta_z^2\in_{\bar{z}_2}^2+\beta_2\delta_z P\in_{\bar{x}_1}\in_{\bar{z}_2}-\\ &\beta_2\delta_z P\in_{\bar{x}_2}\in_{\bar{z}_1}+\psi_3 P\in_{\bar{x}_1}^2-\psi_4 P\in_{\bar{x}_2}^2+(\psi_4-\psi_3)P\in_{\bar{x}_1}\in_{\bar{x}_2}+\beta_2\delta_z\in_{\bar{y}_2}\in_{\bar{z}_1}-P\in_{\bar{y}_2}\in_{\bar{x}_1}+P\in_{\bar{y}_2}\in_{\bar{x}_2}\\ &-\beta_2^2\delta_z^2\in_{\bar{z}_1}\in_{\bar{z}_2}+\frac{\beta_2(\beta_2-1)}{2}\delta_z^2\in_{\bar{z}_1}^2-\beta_2\delta_z P\in_{\bar{z}_2}\in_{\bar{z}_1}+\beta_2\delta_z P\in_{x_2}\in_{\bar{z}_1}-\beta_2\delta_z\in_{\bar{y}_2}\in_{\bar{z}_2}\Bigg]\end{aligned} \tag{3.28}$$

Taking expectation of equation (3.28) and apply the results in equation (3.1), the bias of the suggested estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ when $S_2\subset S_1$ is obtained as

$$\begin{aligned}Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I=\bar{Y}\Bigg[&\theta_2 PC_{yz}C_z^2-\theta_1 PC_{yx}C_x^2-\theta_3\psi_4 PC_x^2-\theta_3\beta_2\delta_z C_{yz}C_z^2\\ &+\theta_2\frac{\beta_2(\beta_2+1)}{2}\delta_z^2 C_z^2+\theta_1\delta_z^2 C_z^2\left(\frac{\beta_2(\beta_2-1)}{2}-\beta_2^2\right)\Bigg]\end{aligned} \tag{3.29}$$

Also, taking expectation of (3.28) and using results of (3.2), the bias of the suggested estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ to terms of order n^{-1} when $S_2\subset\Omega_N$ is obtained is obtained as:

$$\begin{aligned}Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II}=\bar{Y}\Bigg[&\frac{\beta_2(\beta_2+1)}{2}\delta_z^2\theta_2 C_z^2+\frac{\beta_2(\beta_2-1)}{2}\delta_z^2\theta_1 C_z^2-\beta_2\delta_z P\theta_2 C_{xz}C_z^2\\ &+\psi_3 P\theta_1 C_x^2-\psi_4 P\theta_2 C_x^2+P\theta_2 C_x^2 C_{yx}-\beta_2\delta_z\theta_2 C_{yz}C_z^2\Bigg]\end{aligned} \tag{3.30}$$

Square both sides of (3.28), taking expectation and using the results in equation (3.1), we obtain the MSE of the suggested estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case I as:

$$MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2 P\left(P - 2C_{yx}\right) + \theta_3 \beta_2 \delta_z C_z^2\left(\beta_1 \delta_z - 2C_{yz} - 2PC_{xz}\right)\right] \tag{3.31}$$

Also, square both sides of (3.28), taking expectation and using the results in equation (3.2), we obtain the MSE of the suggested estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case II as:

$$\begin{aligned} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}^2\Big[&\theta_2 C_y^2 + \theta_2 C_x^2 P\left(P + 2C_{yx}\right) + \theta_1 \beta_2 \delta_z C_z^2\left(\beta_2 \delta_z - 2PC_{xz}\right) \\ &+ \theta_2 \delta_z \beta_2 C_z^2\left(\delta_z \beta_2 - 2C_{yz} - 2PC_{xz}\right) + \theta_1 P^2 C_x^2\Big] \end{aligned} \tag{3.32}$$

Differentiate equation (3.31) with respect to P and equate to zero,

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}^2\left[2P\theta_3 C_x^2 - 2\theta_3 \rho_{xy} C_x C_y - 2\theta_3 \beta_2 \delta_z \rho_{xz} C_x C_z\right] = 0 \tag{3.33}$$

$$P = \left[\rho_{xy} C_y + \beta_2 \delta_z \rho_{xz} C_z\right] / C_x \tag{3.34}$$

Differentiate equation (3.31) with respect to β_2 and equate to zero,

$$\frac{\partial}{\partial \beta_2} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}^2 \theta_3 \delta_z C_z^2\left[2\beta_2 \delta_z - 2C_{yz} - 2PC_{xz}\right] = 0 \tag{3.35}$$

$$\beta_2 = \left[C_{yz} + PC_{xz}\right] / \delta_z \tag{3.36}$$

Solving P and β_1 simultaneously, the optimum values of P and β_1 are

$$P = \left[\rho_{xy} C_y + \rho_{xz} C_z C_{yz}\right] / \left[C_x - \rho_{xz} C_z C_{xz}\right] = P_{41}^{opt} \tag{3.37}$$

$$\beta_2 = \left[C_x C_{yz} + \rho_{xy} C_y C_{xz}\right] / \delta_z\left[C_x - \rho_{xz} C_z C_{xz}\right] = \beta_{21}^{opt} \tag{3.38}$$

Substituting the values of P_{41}^{opt} and β_{21}^{opt} in equation (3.31) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_2(d)}$ under case I as

$$MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}^2 C_y^2\left[\theta_2 - \theta_3\left(\rho_{xy}^2 + \rho_{yz}^2 + 2\rho_{xy}\rho_{xz}\rho_{yz}\right) / \left(1 - \rho_{xz}^2\right)\right] \tag{3.39}$$

In order to estimate unknown constant $d \in \Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case I, $P = \left[\rho_{xy} C_y + \rho_{xz} C_z C_{yz}\right] / \left[C_x - \rho_{xz} C_z C_{xz}\right]$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = \left[\rho_{xy} C_y + \rho_{xz} C_z C_{yz}\right] / \left[C_x - \rho_{xz} C_z C_{xz}\right] \tag{3.40}$$

$$\frac{fB - C}{A + C + fB} = \left[\rho_{xy} C_y + \rho_{xz} C_z C_{yz}\right] / \left[C_x - \rho_{xz} C_z C_{xz}\right] \tag{3.41}$$

$$\begin{aligned} (m-1)d^3 + (fm + f - 8m + 9)d^2 - (5fm + 5f - 23m + 26)d \\ + (4fm + 4f - 22m + 24) = 0 \end{aligned} \tag{3.42}$$

where $m = \left[\rho_{xy} C_y + \rho_{xz} C_z C_{yz}\right] / \left[C_x - \rho_{xz} C_z C_{xz}\right]$

By solving (3.42), at most 3 zeros d_1, d_2 and d_3 of the polynomials for which (2.2) is optimal under case I will be obtained.
Also, differentiate equation (3.32) with respect to P and β_2 and equate to zero, we obtain:

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}^2\left[2C_x^2\left(\theta_4 P + \theta_2 C_{yx}\right) - 2\theta_4\beta_2\delta_z C_z^2 C_{xz}\right] = 0 \tag{3.43}$$

$$P = \frac{\theta_4\beta_2\delta_z C_z^2 C_{xz} + \theta_2 C_x^2 C_{yx}}{\theta_4 C_x^2} \tag{3.44}$$

$$\frac{\partial}{\partial \beta_2} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}^2\left[2\theta_3\beta_2\delta_z^2 C_z^2 - 2\theta_3 P\delta_z C_z^2 C_{xz} - 2\theta_2\delta_z C_z^2 C_{xz}\right] = 0 \tag{3.45}$$

$$\beta_2 = \left[\theta_2 C_{xz} + \theta_4 P C_{xz}\right] / \theta_4\delta_z \tag{3.46}$$

Solving P and β_2 simultaneously, the optimum values of P and β_2 are

$$P_{42}^{opt} = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right] / \theta_4 C_x\left[1 - \rho_{xz}^2\right] \tag{3.47}$$

$$\beta_{22}^{opt} = \frac{\theta_2\left[C_{yz} - C_{yx}C_{xz}\right]}{\theta_4\delta_z\left(1 - \rho_{xz}^2\right)} \tag{3.48}$$

Substituting the values of P_{42}^{opt} and β_{22}^{opt} in equation (3.32) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_2(d)}$ under case II as

$$\begin{aligned} MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} &= \bar{Y}^2\theta_2\left[C_y^2 + \frac{\left(\rho_{yz}\rho_{xz} - \rho_{xy}C_y\right)}{\theta_4^2\left(1-\rho_{xz}^2\right)^2}\left\{\rho_{yz}\rho_{xz}\theta_2\theta_4\left(1-2C_y\right)\right.\right. \\ &\left. + \theta_4\rho_{xy}C_y\left(2-\theta_2\right) + 2\theta_4\rho_{xz}^2\rho_{xy}C_y\left(\theta_2 - 1\right)\right\} + \frac{\left(C_{yz} - C_{yx}C_{xz}\right)}{\theta_4^2\left(1-\rho_{xz}^2\right)^2}\left\{\theta_2\rho_{yz}C_yC_z\left(\theta_4 - 2\right)\right. \\ &\left.\left. + \theta_2\rho_{xz}C_yC_z\left(2\rho_{yz}\rho_{xz} - \theta_4\rho_{xy}\right)\right\}\right] \end{aligned} \tag{3.49}$$

In order to estimate unknown constant $d \in \Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case II, $P = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right]/\theta_4 C_x\left[1-\rho_{xz}^2\right]$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right] / \theta_4 C_x\left[1 - \rho_{xz}^2\right] \tag{3.50}$$

$$\frac{fB - C}{A + C + fB} = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right] / \theta_4 C_x\left[1 - \rho_{xz}^2\right] \tag{3.51}$$

$$\begin{aligned}(q-1)d^3 + (fq + f - 8q + 9)d^2 - (5fq + 5f - 23q + 26)d \\ + (4fq + 4f - 22q + 24) = 0\end{aligned} \tag{3.52}$$

where $q = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right]/\theta_4 C_x\left[1-\rho_{xz}^2\right]$
By solving (3.52), at most 3 zeros d_1, d_2 and d_3 of the polynomials for which (3.32) is optimal under case II will be obtained.

4.0 EFFICIENCY COMPARISONS

In this section, the efficiency of suggested estimators are compared theoretically with that of some existing estimators in the literature and the conditions for which the suggested estimators performed better have been established.

4.1 Efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}$ over some related estimators

1. The efficiency of suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator $\bar{y}_{RP}^{(dc)}$ suggested by Choudhury and Singh ([CS12]) are compared as,

$$MSE\left(\bar{y}_{RP}^{(dc)}\right)_{I\min} - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{I\min} > 0 \tag{4.1}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 - \frac{\left(\theta_3 C_x^2 C_{yx} + \theta_1 C_z^2 C_{yz}\right)^2}{\left(\theta_3 C_x^2 + \theta_1 C_z^2\right)}\right] - \bar{Y}^2\left[\theta_2 C_y^2 - \theta_3 C_x^2 C_{yx}^2 - \theta_1 C_z^2 C_{yz}^2\right] > 0$$

$$\left[C_{yx} - C_{yz}\right]^2 > 0$$

$$\rho_{yz} < \frac{\rho_{xy} C_z}{C_x} \tag{4.2}$$

$$MSE\left(\bar{y}_{RP}^{(dc)}\right)_{II\min} - MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II\min} > 0 \tag{4.3}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 - \frac{\theta_2^2 C_{yx}^2 C_x^4}{\left(\theta_4 C_x^2 + \theta_1 C_z^2\{1 - 2C_{xz}\}\right)}\right] - \bar{Y}^2\left[\theta_2 C_y^2 - \frac{\theta_2^2 C_{yx}^2 C_x^4}{\left(\theta_4 C_x^2 - \theta_1 C_z^2 C_{xz}^2\right)}\right] > 0$$

$$\left[C_{xz} - 1\right]^2 > 0$$

$$\rho_{xz} > \frac{C_z}{C_x} \tag{4.4}$$

2. The efficiency of suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{31} suggested by Chand ([Cha75]) are compared as,

$$MSE\left(t_{31}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \tag{4.5}$$

$$\begin{aligned}\bar{Y}^2&\left[\theta_2 C_y^2 + \theta_3 C_x^2\left(1 - 2C_{yx}\right) + \theta_1 C_z^2\left(1 - 2C_{yz}\right)\right] \\ &- \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P C_x^2\left(P + 2C_{yx}\right) + \theta_2 \beta_1 C_z^2 \delta_z\left(\beta_1 \delta_z - 2C_{yz}\right)\right] > 0\end{aligned}$$

Set $\beta_1 = C_{yz} / \delta_z$ as obtained in section 3.1

$$\left[P + C_{yx}\right]^2 < \frac{\theta_3\left[\rho_{xy} C_y - C_x\right]^2 + \theta_1\left[\rho_{yz} C_y - C_z\right]^2}{\theta_3 C_x^2}$$

$$P < \left[\theta_3\left[\rho_{xy} C_y - C_x\right]^2 + \theta_1\left[\rho_{yz} C_y - C_z\right]^2 / \theta_3 C_x^2\right]^{1/2} - C_{yx} \tag{4.6}$$

3. The efficiency of suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{32} suggested by Singh and Upadhayaya ([SU01]) are compared as,

$$MSE\left(t_{32}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \tag{4.7}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2\left(1-2C_{yx}\right) + \theta_1 \frac{\bar{Z}}{\bar{Z}+C_z} C_z^2\left(\frac{\bar{Z}}{\bar{Z}+C_z} - 2C_{yz}\right)\right]$$
$$-\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P C_x^2\left(P+2C_{yx}\right) + \theta_2\beta_1 C_z^2\delta_z\left(\beta_1\delta_z - 2C_{yz}\right)\right] > 0$$

Set $\beta_1 = C_{yz}/\delta_z$ as obtained in section 3.1

$$\left[P+C_{yx}\right]^2 < \frac{\theta_3\left[\rho_{xy}C_y - C_x\right]^2 + \theta_1\left[\rho_{yz}C_y - \frac{\bar{Z}}{\bar{Z}+C_z}C_z\right]^2}{\theta_3 C_x^2}$$

$$P < \left\{\left[\theta_3\left[\rho_{xy}C_y - C_x\right]^2 + \theta_1\left[\rho_{yz}C_y - \bar{Z}C_z/\left(\bar{Z}+C_z\right)\right]^2/\theta_3 C_x^2\right]^{1/2} - C_{yx}\right\} \tag{4.8}$$

4. The efficiency of suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{33} suggested by Singh et. al ([SCS07]) are compared as,

$$MSE\left(t_{33}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \tag{4.9}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2\left(1-2C_{yx}\right) + \theta_1 \frac{\bar{Z}}{\bar{Z}+\rho_{xz}} C_z^2\left(\frac{\bar{Z}}{\bar{Z}+\rho_{xz}} - 2C_{yz}\right)\right]$$
$$-\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P C_x^2\left(P+2C_{yx}\right) + \theta_2\beta_1 C_z^2\delta_z\left(\beta_1\delta_z - 2C_{yz}\right)\right] > 0$$

Set $\beta_1 = C_{yz}/\delta_z$ as obtained in section 3.1

$$\left[P+C_{yx}\right]^2 < \frac{\theta_3\left[\rho_{xy}C_y - C_x\right]^2 + \theta_1\left[\rho_{yz}C_y - \frac{\bar{Z}}{\bar{Z}+\rho_{xz}}C_z\right]^2}{\theta_3 C_x^2}$$

$$P < \left\{\left[\theta_3\left[\rho_{xy}C_y - C_x\right]^2 + \theta_1\left[\rho_{yz}C_y - \bar{Z}C_z/\left(\bar{Z}+\rho_{xz}\right)\right]^2/\theta_3 C_x^2\right]^{1/2} - C_{yx}\right\} \tag{4.10}$$

5. The efficiency of suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{34} suggested by Upadhayaya and Singh ([SU01]) are compared as,

$$MSE\left(t_{34}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \tag{4.11}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2\left(1-2C_{yx}\right) + \theta_1 \frac{C_z\bar{Z}}{C_z\bar{Z}+\beta_{2(z)}} C_z^2\left(\frac{C_z\bar{Z}}{C_z\bar{Z}+\beta_{2(z)}} - 2C_{yz}\right)\right]$$
$$-\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P C_x^2\left(P+2C_{yx}\right) + \theta_2\beta_1 C_z^2\delta_z\left(\beta_1\delta_z - 2C_{yz}\right)\right] > 0$$

Set $\beta_1 = C_{yz}/\delta_z$ as obtained in section 3.1

$$\left[P+C_{yx}\right]^2<\frac{\theta_3\left[\rho_{xy}C_y-C_x\right]^2+\theta_1\left[\rho_{yz}C_y-\dfrac{C_z\bar{Z}}{C_z\bar{Z}+\beta_{2(z)}}C_z\right]^2}{\theta_3C_x^2}$$

$$P<\left[\theta_3\left[\rho_{xy}C_y-C_x\right]^2+\theta_1\left[\rho_{yz}C_y-\frac{C_z\bar{Z}}{C_z\bar{Z}+\beta_{2(z)}}C_z\right]^2/\theta_3C_x^2\right]^{1/2}-C_{yx} \tag{4.12}$$

6. The efficiency of suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{35} suggested by Singh ([Sin01]) are compared as,

$$MSE\left(t_{35}\right)_I-MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I>0 \tag{4.13}$$

$$\bar{Y}^2\left[\theta_2C_y^2+\theta_3C_x^2\left(1-2C_{yx}\right)+\theta_1\frac{\beta_{1(z)}\bar{Z}}{\beta_{1(z)}\bar{Z}+\sigma_z}C_z^2\left(\frac{\beta_{1(z)}\bar{Z}}{\beta_{1(z)}\bar{Z}+\sigma_z}-2C_{yz}\right)\right]$$
$$-\bar{Y}^2\left[\theta_2C_y^2+\theta_3PC_x^2\left(P+2C_{yx}\right)+\theta_2\beta_1C_z^2\delta_z\left(\beta_1\delta_z-2C_{yz}\right)\right]>0$$

Set $\beta_1=C_{yz}/\delta_z$ as obtained in section 3.1

$$\left[P+C_{yx}\right]^2<\frac{\theta_3\left[\rho_{xy}C_y-C_x\right]^2+\theta_1\left[\rho_{yz}C_y-\dfrac{\beta_{1(z)}\bar{Z}}{\beta_{1(z)}\bar{Z}+\sigma_z}C_z\right]^2}{\theta_3C_x^2}$$

$$P<\left\{\left[\theta_3\left[\rho_{xy}C_y-C_x\right]^2+\theta_1\left[\rho_{yz}C_y-\beta_{1(z)}\bar{Z}C_z/\left(\beta_{1(z)}\bar{Z}+\sigma_z\right)\right]^2/\theta_3C_x^2\right]^{1/2}-C_{yx}\right\} \tag{4.14}$$

4.2 Efficiency of $\bar{y}_{FTAA}^{\beta_2(d)}$ over some related estimators

1. The efficiency of suggested estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ and suggested estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ are compared as,

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{I\min}-MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{I\min}>0 \tag{4.15}$$

$$\bar{Y}^2C_y^2\left\{\left[\theta_2-\theta_3\rho_{xy}^2-\theta_1\rho_{yz}^2\right]-\left[\theta_2-\frac{\theta_3\left[\rho_{xy}^2+\rho_{yz}^2+2\rho_{xy}\rho_{xz}\rho_{yz}\right]}{1-\rho_{xz}^2}\right]\right\}>0$$

$$\theta_3\left[\rho_{yz}+\rho_{xy}\rho_{xz}\right]^2-\theta\left[1-\rho_{xz}^2\right]>0$$

$$\frac{\rho_{xy}\rho_{xz}}{\rho_{yz}}>\left[\frac{\theta_1}{\theta_3}\left(1-\rho_{xz}^2\right)\right]^{1/2}-1$$

$$\rho_{yz}>\rho_{xy}\rho_{xz}/\left\{\left[\frac{\theta_1}{\theta_3}\left(1-\rho_{xz}^2\right)\right]^{1/2}-1\right\} \tag{4.16}$$

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II\min}-MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II\min}>0 \tag{4.17}$$

$$\bar{Y}^2C_y^2\left[\left(1-2C_y\right)\theta_2\rho_{xz}\rho_{yz}+\left(2-\theta_2\right)\rho_{xy}C_y+2\left(\theta_2-1\right)\rho_{xz}^2\rho_{xy}C_y\right]>0$$

$$\rho_{yz} > \frac{-\left[\left(2-\theta_2\right)\rho_{xy}C_y + 2\left(\theta_2-1\right)\rho_{xz}^2\rho_{xy}C_y\right]}{\left(1-2C_y\right)\theta_2\rho_{xz}}$$

$$\rho_{yz} < \frac{\rho_{xy}C_y\left[2\left(1-\rho_{xy}^2\right)-\theta_2\left(1-2\rho_{xy}^2\right)\right]}{\left(2C_y-1\right)\theta_2\rho_{xz}} \tag{4.18}$$

5.0 EMPIRICAL STUDY

In this section, relative efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ with respect to some existing related ratio and product estimators were investigated using six data sets.

Data 1: Akkus ([Akk16])
y: The amount of produced table olive (tons) , x: The number of fruit trees in that age
z: Collective areas of fruit (decar)

$$N = 287;\ n_1 = 165;\ n_2 = 106;\ \bar{Y} = 1306.62;\ \bar{X} = 124081.969;\ \bar{Z} = 7672.369;$$
$$C_y = 2.26099;\ C_x = 3.51001;\ C_z = 3.51944; \rho_{yx} = 0.8045; \rho_{yz} = 0.8188, \rho_{xz} = 0.965$$
$$\beta_1(x) = 9.3154,\ \beta_2(x) = 110.140;\ \beta_1(z) = 11.382;\ \beta_2(z) = 9.3154;$$

Data 2: Anderson ([And58])
Y: Head length of second son , **Z:** Head breadth of first son , **X:** Head length of first son

$$N = 25;\ n_1 = 10;\ n_2 = 7;\ \bar{Y} = 183.34;\ \bar{X} = 185.72;\ \bar{Z} = 151.12;$$
$$\rho_{xy} = 0.7108;\ \rho_{yz} = 0.6932;\ \rho_{xz} = 0.7346;\ C_y = 0.0546;\ C_x = 0.2422;$$
$$C_z = 0.0488; \beta_1(z) = 0.002;\ \beta_2(z) = 2.6519$$

Data 3: Handiquer et. al. ([HDK11])
Y: forest timber volume in cubic meter (Cum) in 0.1 ha sample plot, X: average tree height in the sample plot in meter (m), Z: average crown diameter in the sample plot in meter (m)

$$N = 2500;\ n_2 = 25;\ n_1 = 200,\ \bar{Y} = 4.63;\ \bar{X} = 21.09;\ \bar{Z} = 13.55,\ \rho_{xy} = 0.79;$$
$$C_y = 0.95;\ C_x = 0.98;\ C_z = 0.64;\ \rho_{xz} = 0.66;\ \rho_{yz} = 0.72$$

Data 4: Johnson ([Jon72])
y: Percentage of hives affected by disease, x: Date of flowering of a particular summer species, z: average January temperature.

$$N = 10;\ n_2 = 4;\ n_1 = 7,\ \bar{Y} = 52;\ \bar{X} = 200;\ \bar{Z} = 42,\ \rho_{xy} = -0.94;$$
$$C_y^2 = 0.0244;\ C_x^2 = 0.0021;\ C_z^2 = 0.017;\ \rho_{xz} = -0.73;\ \rho_{yz} = 0.80$$

Data 5: Singh ([Sin67])
y: Number of female employed, x: Number of educated female, z: Number of female in service

$$N = 61;\ n_2 = 20;\ n_1 = 25,\ \bar{Y} = 7.46;\ \bar{X} = 179.0;\ \bar{Z} = 5.31,\ \rho_{xy} = -0.207;$$
$$C_y^2 = 0.5046;\ C_x^2 = 0.0633;\ C_z^2 = 0.5737;\ \rho_{xz} = -0.0033;\ \rho_{yz} = 0.7737$$

Data 6: Steel and Torrie ([ST60])
y: Log of leaf burn in sec, x: Chlorine percentage, z: Potassium percentage

$N=30;\ n_2=6;\ n_1=14,\ \bar{Y}=0.686;\ \bar{X}=0.8077;\ \bar{Z}=4.654,\ \rho_{xy}=-0.4996;$

$C_y=0.4803;\ C_x=0.7493;\ C_z=0.2295;\ \rho_{xz}=0.4074;\ \rho_{yz}=0.1794$

Table 1: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related estimators

Estimators	Data 1			Data 2			Data 3		
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Case I									
Sample mean	0	5145375	100	0	51873.9	100	0	0.7661	100
Choudhury and Singh ([CS12])	49.70	1814947.1	283.5	0.084	25667.4	202.1	0.018	0.2879	266.1
Chand ([Cha75])	96.88	3961027.7	129.9	0.116	17728.6	292.6	0.024	0.5821	131.6
Singh and Upadhayaya ([SU01])	20.97	4220980.3	121.9	0.147	16610.3	312.3	0.029	0.3047	251.4
Singh et al. ([SCS07])	-17.95	176322.48	291.8	-0.01	26412.4	196.4	-0.00	0.2974	257.6
Singh and Upadhayaya ([SU01])	-36.12	4220980.3	121.9	0.003	16546.7	313.5	-0.08	0.3079	248.8
Singh ([Sin01])	10.83	3396287.1	151.5	-0.02	33273.8	115.9	-0.02	0.3369	227.4
Shukla ([Shu02])	106.87	3716254	138.5	0.009	8.7966	117.2	0.005	0.4446	172.3
Suggested $\bar{y}_{FTAA}^{\beta_1(d)}$	19.65	1619060.7	317.8	0.042	14555.0	356.4	-0.01	0.2664	287.6
Suggested $\bar{y}_{FTAA}^{\beta_2(d)}$	16.82	1720285.9	299.1	0.009	15591.8	332.7	0.005	0.2828	270.9
Case II									
Sample mean	0	5145375	100	0	51873.8	100	0	0.7661	100
Choudhury and Singh ([CS12])	49.64	29570112.1	174.0	0.004	15829.7	327.7	0.019	0.2879	266.1
Chand ([Cha75])	68.41	4529379.4	113.6	0.302	32199.8	161.1	0.037	0.5823	131.9
Singh and Upadhayaya ([SU01])	63.55	4664891.2	110.3	0.431	39568.2	131.1	0.045	0.3976	192.7
Singh et al. ([SCS07])	71.51	3127887.5	164.5	0.547	13803.6	375.8	0.051	0.2975	257.8
Singh and Upadhayaya ([SU01])	-29.34	4751038.78	108.3	0.013	28424.1	182.5	-0.06	0.3479	220.2
Singh ([Sin01])	-39.68	2695324.8	190.9	0.033	35505.7	146.1	-0.02	0.4579	167.3
Shukla ([Shu02])	-39.67	3078888	167.1	0.041	8.8952	115.9	-0.02	0.4950	154.8
Suggested $\bar{y}_{FTAA}^{\beta_1(d)}$	45.53	2231298.8	230.6	0.009	13089.6	396.3	-0.01	0.2823	271.4
Suggested $\bar{y}_{FTAA}^{\beta_2(d)}$	62.25	2588216.9	198.8	0.263	12881.5	402.7	0.001	0.2501	306.4

Table 1 show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and other existing related ratio estimators under two-phase simple random sampling scheme when the study and auxiliary variables are positively correlated. The biases and MSEs are obtained using Data 1, 2 and 3 respectively. These properties (Bias, MSE and PRE) were computed under cases I and II. The results of the analysis revealed that $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ have minimum MSE and high PRE.

Table 2 show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and other related ratio estimators under two-phase simple random sampling scheme when the study and auxiliary variables are negatively correlated using Data 4, 5 and 6 respectively. These properties (Bias, MSE and PRE) were computed under cases I and II. The results of the analysis revealed that $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ have minimum MSEs and high PRE.

Table 2: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related estimators

Estimators	Data 4			Data 5			Data 6		
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Case I									
Sample mean	0	9.8966	100	0	0.9437	100	0	0.0144	100
Choudhury and Singh ([CS12])	0.040	5.2335	189.1	0.044	0.9031	104.5	0.138	0.0108	133.3
Chand ([Cha75])	0.043	7.3526	134.6	0.105	1.4408	65.5	0.141	0.0094	152.9
Singh and Upadhayaya ([SU01])	0.045	7.4579	132.7	0.106	0.7729	122.1	0.144	0.0128	112.4
Singh et al. ([SCS07])	0.001	6.4389	153.7	0.067	0.5350	176.4	0.101	0.0117	123.1
Singh and Upadhayaya ([SU01])	0.007	7.5145	131.7	0.107	0.4921	191.8	0.003	0.0128	112.8
Singh ([Sin01])	0.032	9.1129	108.6	0.117	0.6233	151.4	0.021	0.0114	126.0
Shukla ([Shu02])	0.007	6.645	148.9	0.0007	0.7534	125.3	0.005	0.0111	130.9
Proposed $\bar{y}_{FTAA}^{\beta_1(d)}$	0.039	5.0596	195.6	0.204	0.4815	196.4	0.026	0.0093	155.1
Proposed $\bar{y}_{FTAA}^{\beta_2(d)}$	0.042	4.6419	213.2	0.215	0.5338	176.8	0.028	0.0089	161.7
Case II									
Sample mean	0	9.8966	100	0	0.9437	100	0	0.0144	100
Choudhury and Singh ([CS12])	0.059	6.2204	159.1	0.059	0.8315	113.5	0.051	0.0108	133.2
Chand ([Cha75])	0.063	7.3745	134.2	0.063	0.9710	96.3	0.064	0.0094	152.9
Singh and Upadhayaya ([SU01])	0.067	8.9159	111.0	0.067	0.6576	143.5	0.070	0.0171	84.28
Singh et al. ([SCS07])	0.014	7.1223	137.6	0.014	0.5350	176.4	0.074	0.0117	123.1
Singh and Upadhayaya ([SU01])	0.010	5.7074	173.4	0.010	0.4563	206.8	0.002	0.0189	76.3
Singh ([Sin01])	0.048	9.4433	104.8	0.048	0.8156	115.7	0.005	0.0126	113.9
Shukla ([Shu02])	0.011	6.9524	142.3	0.059	0.7797	121.0	0.007	0.0131	110.7
Suggested $\bar{y}_{FTAA}^{\beta_1(d)}$	0.058	4.4260	223.6	0.058	0.4481	210.6	0.035	0.0087	165.2
Suggested $\bar{y}_{FTAA}^{\beta_2(d)}$	0.062	4.8018	206.1	0.062	0.4505	209.5	0.043	0.0091	157.5

Conclusively, the suggested estimators demonstrate high level of efficiency over the existing estimators considered in the study.

REFERENCES

[Akk16] **Akkus O.** - *A review on the theoretical and empirical efficiency comparisons of some ratio and product type mean estimators in two phase sampling scheme*. American Journal of Mathematics and Statistics, 6(1), 18-35, 2016.

[And58] **Anderson T. W.** - *An Introduction to Multivariate Statistical Analysis*. John Wiley & Sons, Inc., New York, 1958.

[Cha75] **Chand L.** - *Some ratio type estimator based on two or more auxiliary variables*. Ph.D.dissertation. Lowa State University. Ames, Lowa (unpublished), 1975.

[CS12] **Choudhury S., Singh B. K.** – *A class of chain ratio-product type estimators with two auxiliary variables under double sampling scheme*. Jour. Of the Korean Stat. Soc. 41, 247-256, 2012.

[HDK11] **Handique B. K., Das G., Kalita M. C.** - *Comparative studies on forest sampling techniques with satellite remote sensing inputs*, Project Report, NESAC, pp.12-40, 2011.

[Jon72] **Johnson J.** - *Econometric Methods*. Second edition, Mc Graw Hill Book co., Tokyo, 1972.

[KSS12] **Khan H., Shouket S., Sanaullah A.** - *A class of improved estimators for estimating population mean regarding partial information in double sampling.* Global journal of Science Frontier research (Mathematics and Decision), 12 (14), 32-44, 2012.

[Shu02] **Shukla D.** - *F-T estimator under two phase sampling*, Metron, 59 (1-2): 253-263, 2002.

[Sin01] **Singh G.N.** - *On the use of transformed auxiliary variable in the estimation of population mean in two phase sampling*. Statistics in Transition. 5(3), 405-416, 2001.

[Sin67] **Singh M. P.** - *Ratio-cum product method of estimation*. Metrika, 12, 34-42, 1967.

[ST60] **Steel R. G. D., Torrie J. H.** - *Principle and Procedures of Statistics*, Mc Graw Hill Book Co., 1960.

[SU01] **Singh G. N., Upadhyaya L. N.** - *An Empirical Study of Modified Ratio Estimators in Two Phase Sampling in Presence of Coefficient of Variation of the Auxiliary Variable*, Statistics in Transition, 5(2): 319-326, 2001.

[SCS07] **Singh R., Chauhan P., Sawan N.** - *A Family of Estimators for Estimating Population Mean Using Known Correlation Coefficient in Two Phase Sampling*. Statistics in Transition, 8(1), 89-96, 2007.

[SUC04] **Singh H. P., Upadhyaya L. N., Chandra P.** - *A general family of estimators for estimating population mean using two auxiliary variables in two-phase sampling*. Statistics in Transition, 6(7): 1055-1077, 2004.