STABILITY ANALYSIS OF MOTHER-TO-CHILD TRANSMISSION OF HIV/AIDS DYNAMIC MODEL WITH TREATMENT

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ABSTRACT: A Mathematical Model of HIV/AIDS with mother – to – child transmission in the presence of treatment was examine in this paper, it ascertain the impact of treated individuals on the transmission dynamics of HIV/AIDS. Equilibrium points of the model system were found, stability analysis and numerical simulation was carried out. The result show that with treatment, the measure of the effect of health policies is realistic and control rate of mother – to – child is effective.
KEYWORDS: HIV Transmission, ARVs, Compartmental model, stability, equilibrium points and Treatment.

1. INTRODUCTION

One of the greatest public health disaster that gives a serious challenges to the global population for long is the HIV/AIDS. It can be transmitted in many ways but most pronounced methods are Mother – to – Child and Heterosexual transmission. Mother – to – Child transmission normally occurred during the pregnancy, child birth or breast feeding which can be influenced by many factors including maternal viral load and the kind of delivery ([Bas05]).
The virus may be transmitted to the new born babies during pregnancy (in utero), labour delivery (through contamination by blood or other fluids during birth) and during breast feeding. Among infected infants who are not breastfed, about two – thirds of the cases of MTCT occur around the time of delivery and the rest during pregnancy (mostly during the last 2 months). In populations where breast feeding is the norm, it accounts for more than one – third of all transmission. Thus the rate of transmission from unifected to infected depends on the health status of the Mother and the conditional probability that an infected Mother will transmit the virus to either the foetus or newborn in utero during or shortly after delivery which is 21 – 43%.
This paper focus on the dynamic modeling of HIV/AIDS with emphasis on the effect of treatment, it is obvious that awareness and public health education programs on HIV/AIDS treatment can contribute towards eradication of the disease. A lot of mathematical and Statistical models have been proposed to describe the population dynamics of HIV/AIDS, many of these models focus on the theoretical study of HIV/AIDS. But recently, interventions into these models have attracted significant attention.
In this study, we evolve the model proposed by Bashiru et at ([BF09]) who considered Mother-to-Child transmission of HIV/AIDS without treatment. In attempt to measure the performance of public health recommendation policies aimed to control the epidemic may not give a reasonable result. We therefore modified the model proposed by Bashiru et al ([BF09]) with the introduction of treatment compartment.

2. MATERIALS AND METHODS

We propose a simple HIV/AIDS model with treatment class. In this model, the sexually mature population is divided into four compartments: the Susceptible, the Infectives (also assumed to be Infectious), the Treatment and the AIDS population whose numbers are denoted by S, I, T and A respectively. The number of total population is denoted by N(t), at any time t i.e N=S+I+T+A. In the model, we assume that the Susceptibles become HIV infected via sexual contacts with infectives which may also lead to birth of infected children. We also assumed that a fraction of newborns are infected at birth and hence are directly recruited into the Infective class with a rate $\gamma\varepsilon$ ([NH06]). We do not consider direct recruitment of other infected persons except through Mother – to – Child transmission only. We also assume that some of the infectives move to join treated class with the rate σ_2 , while others with serious infection directly join the AIDS class with a rate σ_1.

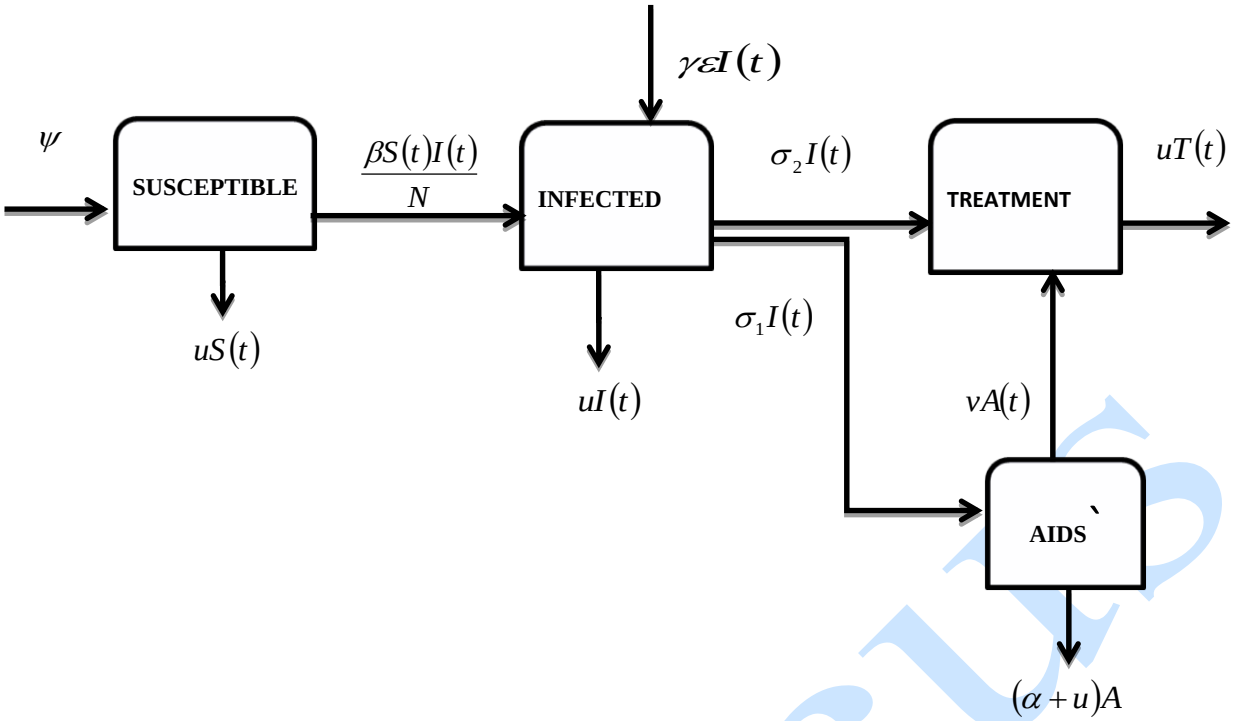

Fig. 1. Proposed schematic diagram for Mother – to – Child Transmission of HIV/AIDS

With the above considerations and assumptions, the spread of the disease is assumed to be governed by the following system of nonlinear ordinary differential equations:

$$\frac{dS}{dt} = \psi - \frac{\beta SI}{N} - uS \tag{1}$$

$$\frac{dI}{dt} = \gamma\varepsilon I + \frac{\beta SI}{N} - uI - \sigma_2 I - \sigma_1 I \tag{2}$$

$$\frac{dT}{dt} = \sigma_2 I - uT + vA \tag{3}$$

$$\frac{dA}{dt} = \sigma_1 I - vA - (\alpha + u)A \tag{4}$$

Where,

α - Disease – induced death rate due to AIDS.

σ_2 - Rate of movement from infectious class to treatment class.

σ_1 - Rate of movement from infectious class to AIDS class.

u - Natural mortality rate.

v - Rate at which AIDS group get treatment

β - The contact rate of the epidemic.

ψ - The rate of recruitment into susceptible population.

γ - Birth rate of infected newborn

ε - The fraction of infected newborn.

3. POSITIVITY AND BOUNDEDNESS OF SOLUTIONS

In this section, the positivity and boundednes of solution of equation 1 to 4 was established as the model described the evolution of population, hence the population should always remain non – negative and bounded; this implies the global existence of the solutions. We assume that the initial data for equation 1 to 4 satisfy

$$S^0 \geq 0,\ I^0 \geq 0, T^0,\ A^0 \geq 0 \tag{5}$$

4. EXISTENCE AND UNIQUENESS OF SOLUTION FOR THE MODEL

For the model to predict the future of the system from its current state at time t_0, the initial value problem (IVP)

$$x' = f(t,x), \qquad x(t_0) = x_0 \tag{6}$$

must have a solution that exist and also unique.

In this subsection, the conditions for the existence and uniqueness of solution for the model will be established.

Let

$$\begin{aligned}
&\frac{dS}{dt} = \psi - \frac{\beta SI}{N} - uS, && S(t_0) = S_0\\
&\frac{dI}{dt} = \gamma\varepsilon I + \frac{\beta SI}{N} - uI - \sigma_2 I - \sigma_1 I, && I(t_0) = I_0\\
&\frac{dT}{dt} = \sigma_2 I - uT + \nu A, && T(t_0) = T_0\\
&\frac{dA}{dt} = \sigma_1 I - \nu A - (\alpha + u)A, && A(t_0) = A_0
\end{aligned} \quad (7)$$

Theorem 1.
Let

$$D = \{(S,I,T,A,t) \| S - S_0| \le \le a, |I - I_0| \le a, |T - T_0| \le a, |A - A_0| \le a \quad (8)$$

and suppose that $f(t,x)$ satisfies the Lipchitz condition

$$\| f(t,x_1) - f(t,x_2) \| \le k \| \|$$

Then equation (7) has a unique solution.
Proof.

$$\begin{aligned}
\frac{dS}{dt} &= f_1(S,I,T,A) = \psi - \frac{\beta SI}{N} - uS\\
\frac{dI}{dt} &= f_2(S,I,T,A,t) = \gamma\varepsilon I + \frac{\beta SI}{N} - uI - \sigma_2 I - \sigma_1 I\\
\frac{dT}{dt} &= f_3(S,I,T,A,t) = \sigma_2 I - uT + \nu A\\
\frac{dA}{dt} &= f_4(S,I,T,A,t) = \sigma_1 I - \nu A - (\alpha + u)A
\end{aligned} \quad (9)$$

Then ;

$$\left.\frac{\partial f_1}{\partial S}\right|_{(S_0,I_0,T_0,A_0)} = \left| -\left(\frac{\beta I}{N} + u\right)\right| = \frac{\beta I}{N} + u,$$

$$\left.\frac{\partial f_1}{\partial I}\right|_{(S_0,I_0,T_0,A_0)} = \left| -\frac{\beta S}{N}\right| = \frac{\beta S}{N}$$

$$\left.\frac{\partial f_1}{\partial T}\right|_{(S_0,I_0,T_0,A_0)} = 0,$$

$$\left.\frac{\partial f_1}{\partial A}\right|_{(S_0,I_0,T_0,A_0)} = 0,$$

$$\left.\frac{\partial f_2}{\partial S}\right|_{(S_0,I_0,T_0,A_0)} = \frac{\beta I}{N}$$

$$\left.\frac{\partial f_2}{\partial I}\right|_{(S_0,I_0,T_0,A_0)} = \frac{\beta S}{N} + \gamma\varepsilon - u - \sigma_2 - \sigma_1,$$

$$\left.\frac{\partial f_2}{\partial T}\right|_{(S_0,I_0,T_0,A_0)} = 0,$$

$$\left.\frac{\partial f_2}{\partial A}\right|_{(S_0,I_0,T_0,A_0)} = 0$$

$$\left.\frac{\partial f_3}{\partial S}\right|_{(S_0,I_0,T_0,A_0)} = 0,$$

$$\left.\frac{\partial f_3}{\partial I}\right|_{(S_0,I_0,T_0,A_0)} = \sigma_2,$$

$$\left.\frac{\partial f_3}{\partial T}\right|_{(S_0,I_0,T_0,A_0)} = |-u| = u,$$

$$\left.\frac{\partial f_3}{\partial A}\right|_{(S_0,I_0,T_0,A_0)} = V$$

$$\left.\frac{\partial f_4}{\partial S}\right|_{(S_0,I_0,T_0,A_0)} = 0,$$

$$\left.\frac{\partial f_4}{\partial I}\right|_{(S_0,I_0,T_0,A_0)} = \sigma_1,$$

$$\left.\frac{\partial f_4}{\partial T}\right|_{(S_0,I_0,T_0,A_0)} = 0,$$

$$\left.\frac{\partial f_4}{\partial A}\right|_{(S_0,I_0,T_0,A_0)} = |-(\nu + \alpha + u)| = (\nu + \alpha + u)$$

Hence $\left|\frac{\partial f_i}{\partial x_i}\right|, i, j = 1,2,3$ are continuous and bounded. Therefore, the problem (7) has a unique solution.

5. THE BASIC REPRODUCTIVE NUMBER R_0

The basic reproductive number (R_0) for the model adopt was is calculated using linearization method, linearizing the equation for $\frac{dI}{dt}$ about $I = 0$. Hence, the basic reproductive number (R_0) for the equation (1) to (4) model is

$$R_0 = \frac{\beta\psi + uN\gamma\varepsilon}{(u + \sigma_2 + \sigma_1)uN} \quad (10)$$

6. STABILITY ANALYSIS OF THE MODEL

The equation (1) to (4) exhibits two steady states namely:

$$E_0 = \left(S^0, I^0, T^0, A^0\right) = \left(\frac{\psi}{u}, 0, 0, 0\right) \tag{11}$$

and

$$E^* = \left(S^*, I^*, T^*, A^*\right) = \begin{pmatrix} -\frac{Na}{\beta}, -\frac{R_0 - 1}{(\gamma\varepsilon - \sigma_1 - u - \sigma_2)\beta[uN(\sigma_1 + u + \sigma_2)]}, \\ -\frac{[\sigma_2(\nu + u + \alpha) + \nu\sigma_1](R_0 - 1)}{N\beta(\gamma\varepsilon - \sigma_1 - u - \sigma_2)(\nu + u + \alpha)[(\sigma_1 + u + \sigma_2)]}, \\ -\frac{\sigma_1(R_0 - 1)}{uN\beta(\gamma\varepsilon - \sigma_1 - u - \sigma_2)(\nu + u + \alpha)[(\sigma_1 + u + \sigma_2)]} \end{pmatrix} \tag{12}$$

6.1 Stability of the Disease Free Equilibrium

To examine the local stability of disease – free – equilibrium E_0 we evaluate the jacobian matrix at $E_0 = \left(S^0, I^0, T^0, A^0\right)$.

Considering equation (7) and let $\mathrm{X} = S - S_0$, $I = I$, $T = T$ and $A = A$

The jacobian matrix is

$$Jp\left(E_0\right) = \begin{pmatrix} -u & -\frac{\beta S_0}{N} & 0 & 0 \\ 0 & \frac{\beta S_0}{N} - (u + \sigma_2 + \sigma_1 - \gamma\varepsilon) & 0 & 0 \\ 0 & \sigma_2 & -u & \nu \\ 0 & \sigma_1 & 0 & -(\nu + \alpha + u) \end{pmatrix} \tag{13}$$

Preposition: E_0 is local asymptotically stable if $R_0(\lambda) < 0$

Proof.

$$\left|J\left(E_0\right) - \lambda I\right| = \begin{pmatrix} -(u + \lambda) & -\frac{\beta\psi}{uN} & 0 & 0 \\ 0 & (\frac{\beta\psi}{uN} - (u + \sigma_2 + \sigma_1 - \gamma\varepsilon)) - \lambda & 0 & 0 \\ 0 & \sigma_2 & -(u + \lambda) & \nu \\ 0 & \sigma_1 & 0 & -(\nu + \alpha + u + \lambda) \end{pmatrix} = 0 \tag{14}$$

The eigenvalues of $J\left(E_0\right)$ are

$$-(u + \lambda)\left[\left(\frac{\beta\psi}{uN} - (u + \sigma_2 + \sigma_1 - \gamma\varepsilon) - \lambda\right)(u + \lambda)(\nu + \alpha + u + \lambda) - 0\right] = 0 \tag{15}$$

$$\lambda_1 = -u\ , \quad \lambda_2 = \frac{\beta\psi}{uN} - u - \sigma_2 - \sigma_1 + \gamma\varepsilon, \quad \lambda_3 = -u\ , \quad and \quad \lambda_4 = -(\nu + \alpha + u) \tag{16}$$

It can be seen that ;

$$\frac{\beta\psi}{uN} - u - \sigma_2 - \sigma_1 + \gamma\varepsilon < 0$$
$$\frac{\beta\psi + uN\gamma\varepsilon}{(u+\sigma_2+\sigma_1)uN} < 1 \tag{17}$$

Therefore $R_e(\lambda) < 0$ since all the parameters are non – negative. This proves the preposition.

6.2 Stability of Endemic Equilibrium

To examine the local stability of the endemic equilibrium E_*, we evaluate the jacobian matrix at

$$S_* = -\frac{Na}{\beta} \tag{18}$$

$$I_* = -\frac{R_0 - 1}{(\gamma\varepsilon - \sigma_1 - u - \sigma_2)\beta[uN(\sigma_1 + u + \sigma_2)]} \tag{19}$$

$$T_* = -\frac{[\sigma_2(v+u+\alpha) + v\sigma_1](R_0 - 1)}{N\beta(\gamma\varepsilon - \sigma_1 - u - \sigma_2)(v+u+\alpha)[(\sigma_1 + u + \sigma_2)]} \tag{20}$$

$$A_* = -\frac{\sigma_1(R_0 - 1)}{uN\beta(\gamma\varepsilon - \sigma_1 - u - \sigma_2)(v+u+\alpha)[(\sigma_1 + u + \sigma_2)]} \tag{21}$$

Where,

$$a = \gamma\varepsilon - \sigma_2 - u - \sigma_1$$

$$J(E_*) = \begin{pmatrix} -(\frac{\beta I_*}{N} + u) & -\frac{\beta S_*}{N} & 0 & 0 \\ \frac{\beta I_*}{N} & \frac{\beta S_*}{N} - (u + \sigma_2 + \sigma_1 - \gamma\varepsilon) & 0 & 0 \\ 0 & \sigma_2 & -\mu & v \\ 0 & \sigma_1 & 0 & -(v + \alpha + u) \end{pmatrix} \tag{22}$$

The resulting jacobian matrix is

$$|J(E_*) - \lambda I| = \begin{pmatrix} -(\frac{\beta I_*}{N} + u + \lambda) & -\frac{\beta S_*}{N} & 0 & 0 \\ \frac{\beta I_*}{N} & \frac{\beta S_*}{N} - (u + \sigma_2 + \sigma_1 - \gamma\varepsilon + \lambda) & 0 & 0 \\ 0 & \sigma_2 & -(u+\lambda) & v \\ 0 & \sigma_1 & 0 & -(v + \alpha + u + \lambda) \end{pmatrix} = 0 \tag{23}$$

Therefore we obtained the solution of equation (23) as,

$$\lambda^4 - \lambda^3 a_1 + \lambda^2 a_2 + \lambda a_3 + a_4 \tag{24}$$

where,

$$a_1 = \gamma\varepsilon - 2u - v - \alpha - \sigma_2 + \sigma_1 - \frac{\beta S_*}{N} - \beta I_* \tag{25}$$

$$a_2 = u\gamma\varepsilon + u\sigma_1 - u\frac{\beta S_*}{N} - u\sigma_2 - u\alpha - uv - u^2 - \left(\sigma_1 + \sigma_2 + \frac{\beta S_*}{N}\right)(2u + v + \alpha) - \beta I_*\left(u + v + \alpha + \sigma_2 - \gamma\varepsilon - \sigma_1 + \frac{\beta S_*}{N}\right) + \frac{\beta S_*}{N}\left[\frac{\beta I_*}{N}\right] \tag{26}$$

$$a_3 = u^3 - u^2(\sigma_2 - v - \alpha - \sigma_1) - uv\sigma_2 - u\alpha\sigma_2 - uv\sigma_1 - u\alpha\sigma_1 - (uv + u\alpha + u^2) + (2u + uv + u\alpha)\left(\sigma_1 + \sigma_2 + \frac{\beta S_*}{N}\right)[1 - \beta I_*] - (2u + v + \alpha)\frac{\beta S_*}{N}\left[\frac{\beta I_*}{N}\right] \tag{27}$$

$$a_4 = -u^4 - u(u + v + \alpha)\frac{\beta S_*}{N}\left[\frac{\beta I_*}{N}\right] + u^3(\sigma_2 - v - \alpha - \sigma_1) + u^2(v\sigma_2 - v\sigma_1 - \alpha + \alpha\sigma_1) + (u^2 v - u^2 + u^3) + u^3\beta I_*(\sigma_2 - v - \alpha - \sigma_1) + \beta I_*\left[uv\sigma_2 + u\alpha + \sigma_2 + \sigma_2 v + u\alpha\sigma_1 + (uv + u\alpha + u^2)\frac{\beta S_*}{N}\right]$$

By Routh-Hurwitz criteria ([LWJ06]) the endemic equilibrium E_* is local asymptotically stable if $a_1 > 0,\ a_2 > 0, a_3 > 0,\ a_4 > 0,\ a_1 a_2 - a_3 > 0$ *and* $a_2 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0$.

7. SENSITIVITY ANALYSIS

A dynamic model is said to be sensitive to a parameter value if there is a big effect on the outcome, due to a small change in the parameter value have otherwise it is insensitive.

7.1 Initial Conditions

The base year used in our simulations was 2010, the population in Nigeria was approximately 160.2 million in 2010. The number of HIV/AIDS –positive individuals in 2010 was 3.46 million. The number of individual receiving ARV treatment was 1.45 million NACA (2012).

7.2 Parameter Values

Constant recruitment rate (ψ) was estimated as the net birth plus the present net immigration. The net birth was calculated using the average of birth in Nigeria, the birth value obtained was adjusted with the infant mortality rate. According to the NDHS 2010, birth rate is 2.7 and mortality rate is 75 per 1000 population (2008 NDHS) therefore survival rate (1 – Mortality rate) is (1- 0.075) which is 0.925. We used the number of people in Nigeria in 2010 which was 162.245 million. Also the net annual migration in to Nigeria was 1.065 million. Therefore, we computed the constant recruitment rate as

ψ = Population x birth rate x infant survival rate +migration ([YB11])

$= 162.245 \text{ X } 0.027 \text{ X } (1 - 0.075) + 1.065$ million

= 5.12 million.

Natural death rate (u). This was estimated using the life expectancy for a Nigerian at birth which was 47.7 years old in 2010 ([Bas14]).

8. RESULTS AND DISCUSSION

Table 1. Assumed Values of Parameters Used in the Data Simulation ([BO15])

Parameters Notation	Parameters	Assumed Values
c	Average number of sexual partner per unit time	1
q	Infant Survival Rate	0.925
p	Proportion of Children that survive after age 15	0.9
β	The contact rate of the epidemic	0.5
ε	The fraction of infected newborn	0.02
γ	Rate of infected newborn baby	0.15
σ_1	Rate of movement from infected class to AIDS	0.04
σ_2	Rate of movement from infected class to Treatment	0.6
δ	Probability of infection from a sexual contact with an infected.	0.4
α	Disease – induced death rate	0.02
w	Sexual contact rate between a sexually mature S person and I person.	0.5

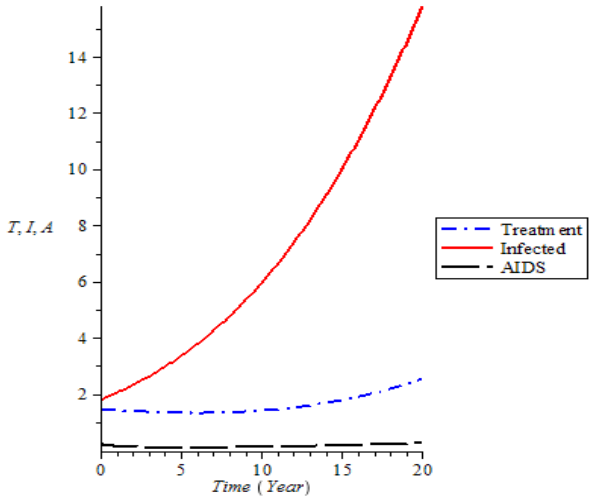

Fig. 2. Population plot of Treatment, AIDS and Infected when $\gamma = 0.05$ and other parameters remain constant

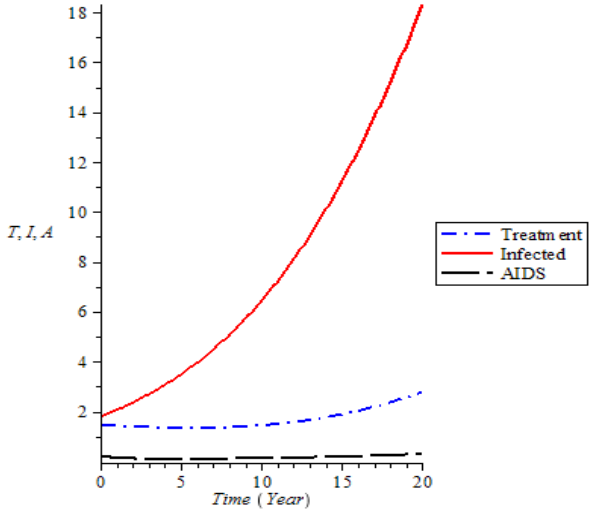

Fig. 3. Population plot of Treatment, AIDS and Infected when $\gamma = 0.15$

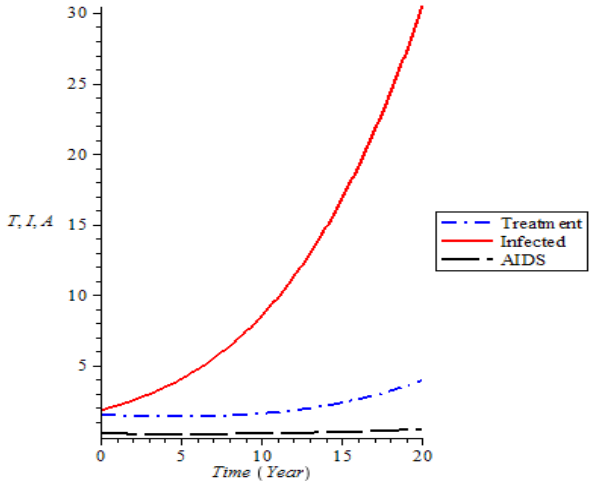

Fig. 4. Population plot of Treatment, AIDS and Infected when $\gamma = 0.5$ and other parameters remain constant

The rate of recruitment of infected children directly into infective class is explicitly shown in Figures 2, 3 and 4. It is seen that as the rate of infected children born γ increase, the infective population also increases. It may be noted here that the birth of infected children make the infective population increase. Also AIDS population decreases with time then it increases until reaches its equilibrium position. Thus, if the birth of infected children is controlled by treatment, the overall infective population will remain under control. This will help in reducing the AIDS population.

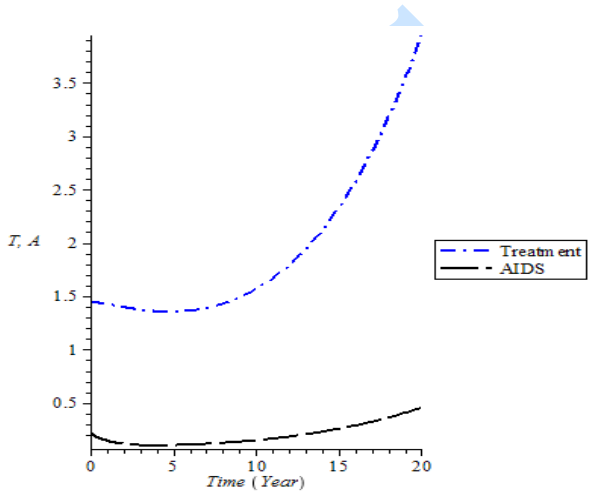

Fig. 5. Population plot of Treatment and AIDS when $\sigma_2 = 0.02$ and other parameters remain constant

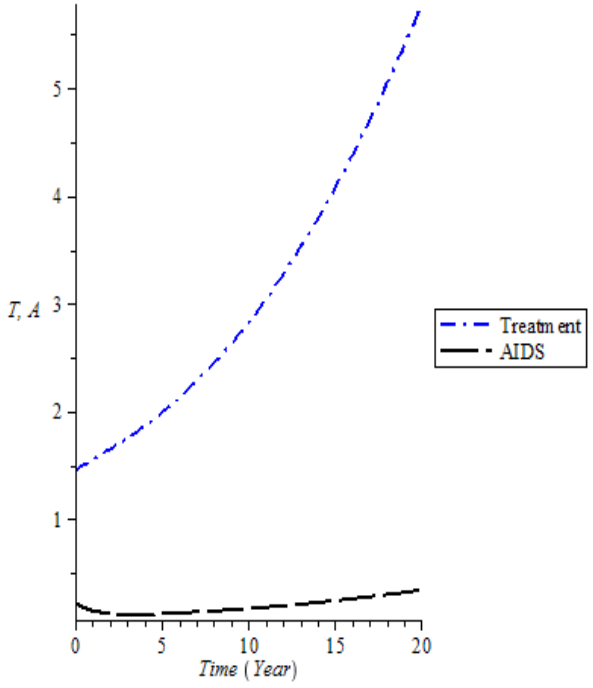

Fig. 6. Population plot of Treatment and AIDS when $\sigma_2 = 0.09$ and other parameters remain constant

It is seen that from the figures 5 and 6 that as σ_2 increases, Treatment population also increases continuously it might be as a result of treatment

which is provided to the patients. This is caused by ARVs it prolonging the life span, then the patient join full blown AIDS, but in AIDS population it decreases continuously up to equilibrium then it increase because disease still exist.

CONCLUSIONS

In this paper, a mathematical model to investigate the effect of treatment on mother – to – child transmission of HIV/AIDS was proposed, the infected babies born joined the infected class directly was assumed as a result of sexual activities between susceptible and infectives. The disease free and endemic equilibrium was obtained and their stabilities.
The numerical investigation of the model was carried out to ascertain the effect of some key parameters on the spread of the disease. It was established that disease free equilibrium is locally asymptotically stable, since $R_0 < 1$. It can also be seen that if the rate of mother – to – child transmission can be control by effective treatment or by creating awareness on HIV/AIDS treatment, it will help to delay the progression to full blown AIDS. Also if there is serious encouragement on uses of condom to put the infectives under control or to resist sexual contact with the infective population.

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