Simulating a journey to Mars with GeoGebra

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ABSTRACT: The main objective of this work is the design and simulation with the math software GeoGebra, of a journey to Mars for a spacecraft launched from Earth, following a Hohmann Transfer Orbit (HOT) to complete the travel. Mars has fascinated mankind since antiquity. We are seeing a resurgence of this interest in the wake of many successful attempts to land on Mars. In this paper we want to harness the power of algebraic and geometric visualizations that GeoGebra offers us to make some constructions of real simulations of the orbit traced by a spacecraft to reach the red planet.
KEYWORDS: GeoGebra, Hohmann Transfer Orbit, planetary motion, orbital parameters.

Introduction

Mars has fascinated mankind since antiquity. The retrograde motion of the red planet provided the impetus for the Earth-centered solar system of Ptolemy, and 1500 years later, for the Sun-centered solar system of Copernicus. We can see in [Tor10] a more detailed description of the retrograde motion of Mars with some GeoGebra constructions, trying to explain the motion of Mars from the Ptolemy's point of view (the theory of epicycles) and from the Copernicus' solar system model.

Kepler (1571 – 1630), in an attempt to improve our Copernicus' model of circular motion, showed that the planet's path was elliptical and not circular, finding his three laws of planetary motion. Kepler's laws provide us an approximate description of the motion of planets around the Sun. Kepler's laws can be resumed as:

First law: The orbit of every planet is an ellipse with the Sun at a focus.

Second law: A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Third law: The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Almost a century later, Isaac Newton proved that relationships like Kepler's would apply exactly under certain ideal conditions approximately fulfilled in the solar system, as consequences of Newton's own laws of motion and law of universal gravitation, using classical Euclidean geometry.

To further explore in the history of ancient astronomy, we can see, among the extensive literature [Aab01, Bru09, Dre53].

Mars has long been considered the most realistic location for life in our solar system, aside from Earth. This has an explanation. Mars captured the attention of the astronomers around 1900, when the American astronomer Percival Lowell, using a new and powerful telescope in Arizona, set his sights in the red planet and concluded that there was a network of waterways on the Mars' surface. This fact represented the proof of the existence of life on Mars.

In July of 1965, the spacecraft Mariner 4 transmitted 22 close-up pictures of Mars. All that was revealed was a surface containing many craters and naturally occurring canals but no evidence of artificial canals or flowing water. Finally, in July and September 1976 Viking landers 1 and 2 touched down on the surface of Mars.

In this paper, our main objective is to describe the path that a spacecraft should follow for a journey to mars, performing a graphical simulation of the orbit with the help of the software GeoGebra.

Recently, Mullins [Mul04] featured a paper in response to the new interest in space exploration that American govern created in 2003 with the support of a program aimed at a human landing on Mars.

1 Preliminary calculations, data and assumptions

Before discussing the mission to Mars we require a few preliminary calculations in preparation for the planning of the trajectories. This background can be found with much more detail in introductory physics textbooks.

The first Kepler's law establishes that the orbit of a planet is an ellipse with the sun at a focus. Symbolically,

$$r = \frac{p}{1 + e\cos\theta} \qquad (1)$$

where *(r, θ)* are the heliocentric polar coordinates for the planet, *p* is the semilactus rectum and *e* is the eccentricity (Fig.1).

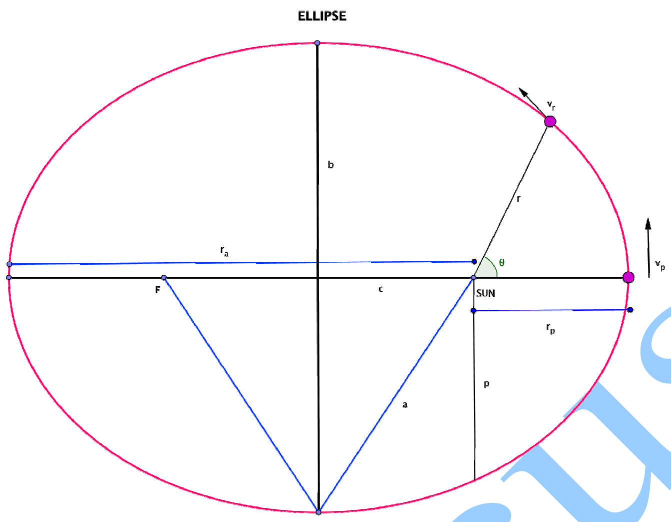

Figure 1. The ellipse and its parameters

At θ = 0°, the minimum distance is $r_{\min} = \dfrac{p}{1+e}$.

At θ = 180°, the maximum distance is $r_{\max} = \dfrac{p}{1-e}$.

The semi major axis is the arithmetic mean between r_{min} and r_{max}, that is $a = \dfrac{r_{\min} + r_{\max}}{2} = \dfrac{p}{1-e^2}$, so, the equation (1) can be written as

$$r = \frac{a(1-e^2)}{1+e\cos\theta}. \quad (2)$$

The semi minor axis is the geometric mean between r_{min} and r_{max}

$$b = \sqrt{r_{\min} r_{\max}} = \frac{p}{\sqrt{1-e^2}}.$$

To find the velocity of a body orbiting about the Sun we use the *vis-viva equation*, based on the total energy of an orbiting body

$$E_{total} = \frac{1}{2}mv^2 - \frac{GM_S m}{r}.$$

This total energy can be shown to be equal to $-\dfrac{GM_S m}{2a}$, with a the semi major axis. Therefore, $v_r = \left[GM_S(2/r - 1/a)\right]^{1/2}$, where G is the universal gravitational constant and M_S is the mass of the sun.

The escape velocity from a planet is obtained by equating the gravitational potential energy from infinity to the surface to the kinetic energy required to overcome that potential energy, that is,

$$\frac{1}{2} m v_{esc}^2 = \frac{G M_{planet} m}{R_{planet}}.$$

Substituting values in this equation we obtain that the escape velocity from earth is 11.2 km s^{-1} and the escape velocity from mars is 5.1 km s^{-1}. When planning our trip to mars we will need to know how far we must be before we can ignore the effects of the gravitational attraction of the planet. That is what we call the body's sphere of influence. Then, the Earth's sphere of influence is about 9.2 x 10^8 m. while the Mars' sphere of influence is about 5.7 x 10^8 m.

For our calculations, we will assume that earth and mars orbits are circular, with radii equal to 1.00 AU and 1.52 AU; therefore, their orbital velocities are constant. Moreover, we can safely consider the two orbits to be coplanar.

The third Kepler's law establishes that the square of the orbital period of a planet is directly proportional to the cube of the semi major axis of its orbit. Symbolically, $P^2 \propto a^3$, where P is the orbital period of the planet and a is the semi major axis of its orbit. The proportional constant is the same for any planet around the sun, so we can write $\frac{P_{planet}^2}{a_{planet}^3} = \frac{P_{earth}^2}{a_{earth}^3}$, so the period of any celestial body can be calculated using the expression

$$P = 365 \cdot a^{3/2} \tag{3}$$

where P is the period in days and a is the semi major axis, given in AU. The periods of the Earth and Mars will be taken as 365 days and 687 days, respectively.

In order to simplify the calculation of the orbital parameters of the trajectory to Mars, we need to assume that when calculating the velocity required when the spacecraft leaves the earth (perigee velocity) and the velocity required to connect with the orbit of Mars, the spacecraft will be outside the sphere of influence of the Earth and Mars, respectively.

2 Hohmann Transfer Orbits (HOT)

In orbital mechanics, the Hohmann Transfer Orbit (HOT) is an orbital maneuver using two engine impulses which, under standard assumptions, move a spacecraft between two coplanar circular orbits. This maneuver was named after Walter Hohmann, the German scientist who published a description of it in 1925, (see [Hoh25]).

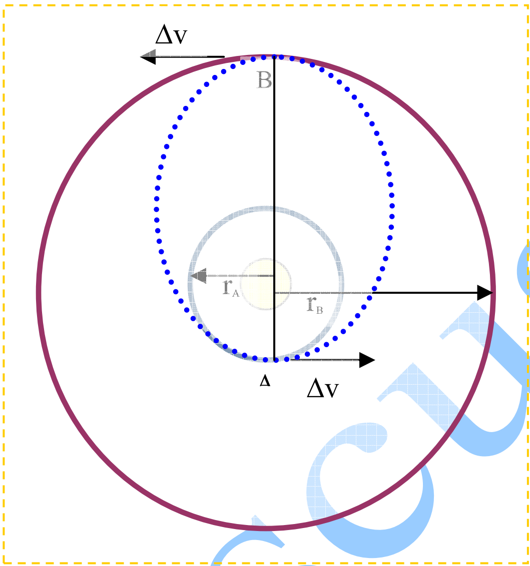

Figure 2. Hohmann transfer orbit

The Hohmann transfer orbit is one half of an elliptic orbit that touches both the orbit that one wishes to leave (point A in the figure 2) and the orbit that one wishes to reach (point B in the figure 2). The transfer is initiated by firing the spacecraft's engine in order to accelerate it so that it will follow the elliptical orbit; this adds energy to the spacecraft's orbit. When the spacecraft has reached its destination orbit, its orbital speed (i.e., orbital energy) must be increased again in order to make its new orbit circular; the engine is fired again to accelerate it to the required velocity.

The Hohmann transfer orbit is theoretically based on impulsive velocity changes to create the circular orbits, therefore a spacecraft using a Hohmann transfer orbit will typically use high thrust engines to minimize the amount of extra fuel required to compensate for the non-impulsive maneuver. Low thrust engines can perform an approximation of a Hohmann transfer orbit, by creating a gradual enlargement of the initial circular orbit through carefully timed engine firings. This requires a delta-v that is up to 141% greater than the 2-impulse transfer orbit, and takes longer to complete.

In astrodynamics, the term delta-v, literally "change in velocity", has a specific meaning: it is a scalar which takes units of speed that measures the amount of "effort" needed to carry out an orbital maneuver, i.e., to change from one trajectory to another.

Hohmann transfer orbits also work to bring a spacecraft from a higher orbit into a lower one; in this case, the spacecraft's engine is fired in the opposite direction to its current path, decelerating the spacecraft and causing it to drop into the lower-energy elliptical transfer orbit. The engine is then fired again in the lower orbit to decelerate the spacecraft into a circular orbit.

When used to move a spacecraft from orbiting one planet to orbiting another, the situation becomes somewhat more complex. For example, consider a spacecraft traveling from the Earth to Mars. At the beginning of its journey, the spacecraft will already have a certain velocity associated with its orbit around Earth – this is velocity that will not need to be found when the spacecraft enters the transfer orbit (around the Sun). At the other end, the spacecraft will need a certain velocity to orbit Mars, which will actually be less than the velocity needed to continue orbiting the Sun in the transfer orbit, let alone attempting to orbit the Sun in a Mars-like orbit.

Therefore, the spacecraft will have to decelerate and allow Mars' gravity to capture it. Therefore, relatively small amounts of thrust at either end of the trip are all that are needed to arrange the transfer. Note, however, that the alignment of the two planets in their orbits is crucial – the destination planet and the spacecraft must arrive at the same point in their respective orbits around the Sun at the same time.

3 The trajectory for a trip to mars

In this section we proceed to calculate the HOT orbit of a spacecraft for a trip to the planet Mars. We already know that the HOT trajectory between two circular (or near-circular) orbits is one of the most useful maneuvers available to satellite operators (see [SB05]).

In addition, transfer orbits of this type can also be used to move from a lower solar orbit to a higher solar orbit, e.g. from the Earth's orbit to that of Mars. The reason to follow a HOT trajectory is because of its characteristics: it is the longest one but the least energy demanding, which constitutes the main objective when designing a trip to the solar system.

Then, the HOT trajectory we are going to follow is the orbit we show in the figure 3.

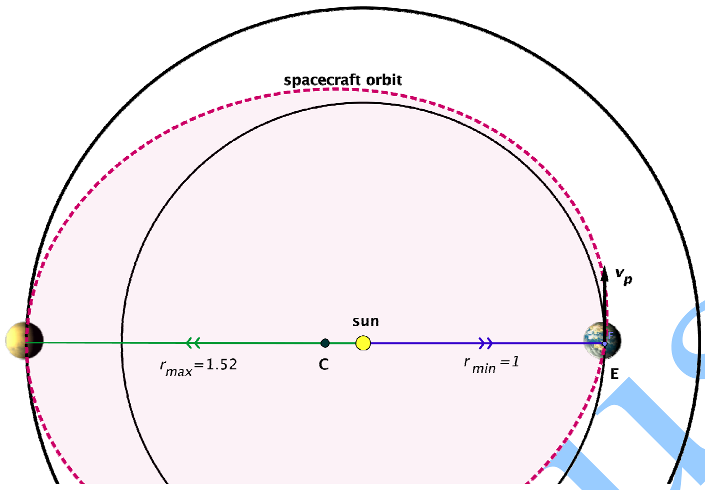

Figure 3. HOT trajectory for a spacecraft traveling from the Earth to Mars

The trajectory is an ellipse with the sun located in one focus and the following parameters: $r_{min} = r_p = 1$, $r_{max} = r_a = 1.52$, where r_p is the perihelion distance and r_a is the aphelion distance. According to the notation in the figure 1, a is the semi major axis, e is the eccentricity of the orbit and v_p and v_a are the perihelion and the aphelion velocities, respectively.

Therefore, the spacecraft must be launched when the earth is in the perihelion and should arrive Mars at the aphelion point. Let us obtain the orbit's elements.

We can compute the semi major axis as

$$a = \frac{r_p + r_a}{2} = \frac{1+1.52}{2} = 1.26.$$

Then, the semi major axis of the ellipse is $a = 1.26$. Now, we compute the eccentricity as

$$e = \frac{c}{a} = \frac{0.26}{1.26} = 0.206.$$

With these parameters, we can determine the equation (in polar coordinates) of the ellipse, that is,

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{1.26(1-0.206^2)}{1+0.206\cos\theta} = \frac{1.206}{1+0.206\cos\theta}.$$

Then,

$$r = \frac{1.206}{1+0.206\cos\theta} \tag{4}$$

describes, with all the assumptions we made in section 2, the trajectory of the spacecraft.

Using expression (3) we can determine the period of the orbit, as $P = 365(1.26)^{3/2} = 516d$, that is 516 days; as a consequence of this period, the time to flight to Mars will be 516/2 = 258 days. Now, using the vis-viva equation to find v_p and v_a, respectively, we have that

$$v_p = \sqrt{GM_S\left(\frac{2}{r_p} - \frac{1}{a}\right)} = \sqrt{GM_S\left(\frac{2}{1} - \frac{1}{1.26}\right)} = 32.7kms^{-1}$$

$$v_p = \sqrt{GM_S\left(\frac{2}{r_a} - \frac{1}{a}\right)} = \sqrt{GM_S\left(\frac{2}{1.52} - \frac{1}{1.26}\right)} = 21.5kms^{-1}$$

Remember that we are assuming that when calculating the perigee and apogee velocities, the spacecraft is beyond the sphere of influence of the Earth and Mars, respectively.

The *Δv* for escaping the earth and being injected into the HOT orbit is *(11.2 + (32.7 - 29.8)) km* s^{-1} *= 14.1 km* s^{-1}, where *11.2* is the escape velocity from Earth, 32.7 is the perigee velocity and 29.8 is the earth orbital velocity.

One more adjustment must be made respect to the position of Mars. It may be demonstrated that the optimal position for Mars in its orbit when the HOT orbit is reached is 44° ahead the position of the Earth, as we can see in the figure 4.

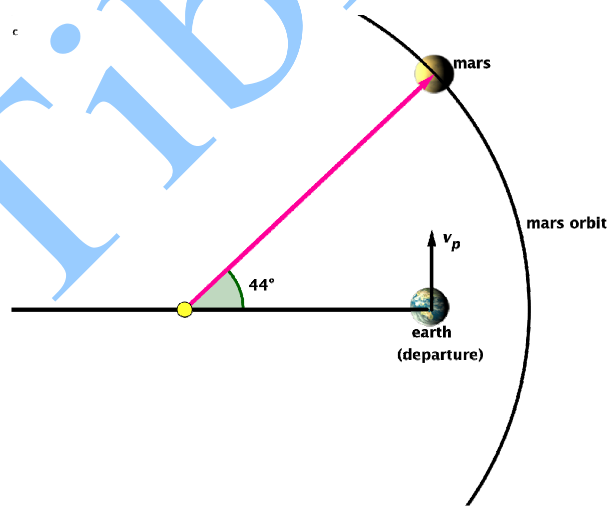

Figure 4. Position of Mars when the spacecraft reaches the HOT orbit

4 The trajectory with GeoGebra

We proceed to develop the simulation of the trip to Mars with GeoGebra in some steps:

1. We draw the orbits for the Earth and Mars, assuming that they are circular (they are near-circular).
2. We represent the elliptic HOT transfer orbit for the spacecraft.
3. Finally, we perform the simulation of the encounter with Mars, synchronizing the trajectories of all the celestial bodies involved.

First we establish the position of the sun in the center of our coordinate system, since all the coordinates refer the sun. Thus, in the editing GeoGebra window we write

Sun=(0,0)

Now let's set the time and, for that purpose, we build a slider, using

We will take the timeline so that $t = 6$ represents an angle of π radians. The slider can be defined as an interval between $t = 0$ and $t = 30$, which means that we run 5π radians. We will also assume that both the Earth's orbit, as Mars are circular.

The orbit of the Earth in polar coordinates can be represented as

Earth=(cos(0.52*t), sin(0.52*t))

To understand the multiplication by the factor 0.52, we must remember that the Earth has an orbital period of 365 days, whereas in our time scale, for $t = 6$ we describe 3.14 radians, what represents a factor of 3.14 / 6 = 0.52. Taking now into account that the radius of orbit of Mars is 1.52 AU and its orbital period is 1.88 years, we can represent the orbit of Mars by the equation (in polar coordinates),

Mars=(1.52; t/1.88 *0.52)

Now we draw the vectors that connect the Earth to the Sun and Mars with the Sun, by using the option  and we label them as u and v, respectively.

Now it is necessary to determine the angle between vectors u and v constructed above; in this way, we always visualize the angle between Earth and Mars in their orbits. We can use the statement GeoGebra option .

So far, we have represented the circular orbits of Earth and Mars around the sun. If we consider $t = 0$, we see that the three celestial bodies are aligned. However, this is not the ideal setting for the HOT transfer orbit, since we know that the optimal geometric configuration is when Mars is ahead of the Earth about 44 degrees, as we see in the figure 4. We know that this 44 degrees represent 0.77 radians, so we modify the initial definition of Mars in this way:

Mars=(1.52; 0.77 + t/1.88 *0.52)

With this new definition, we get the configuration that we need to launch the spacecraft. On this picture we draw the ellipse that represents the orbit of the spacecraft into its trajectory toward the planet. Recall that the equation was given by the expression (4), which constitutes the expression that must be represented.

To represent the equation (4), we develop the following steps:

Step 1: We define the angle θ as a slider, but now is defined as an angle, varying between 0 and 360 degrees.
Step 2: We now define the equation of the ellipse for θ = 0, by

r = 1.206 / (1 + 0.206 * cos(θ))

Note that this point coincides, for θ = 0°, with the point representing the Earth.
Step 3: Now we have to define the spacecraft as a point *Sc*, which polar coordinates are

Sc = (r; θ)

In addition, in the Object Properties of *Sc*, we must mark the option "Show Trace".

Object Properties → Show Trace

With these three steps, we have two sliders, one covering the time and the other related to the spacecraft's orbit. When you move the slider on the angle θ, we see immediately how it is drawing the elliptical orbit of the spacecraft, beginning at the Earth's perigee and reaching the Mars' orbit at the orbit's apogee. We can see what we mean in the figure 5.

So far we have visualized the orbits of Earth and Mars and designed the HOT trajectory of a spacecraft in a trip from the Earth to Mars. However, we have not made any simulation to show that the encounter between the spacecraft and the planet is produced and to confirm the previous calculations.

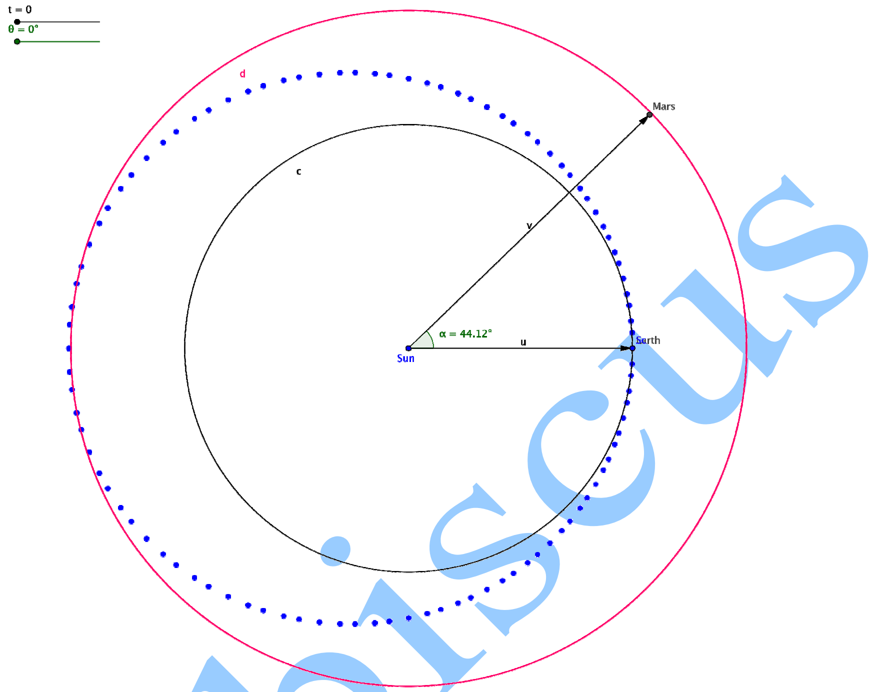

Figure 5. HOT transfer orbit for a spacecraft from the Earth to Mars

To perform the simulation the only thing we should do is to synchronize the two sliders, that is, to synchronize the angle (in radians) over time. We have already established the period of the HOT orbit of the spacecraft, which is $P_{Sc} = 516$ days, which represents a period (in years) of

$P_{Sc} = 516/365 = 1.41$ *years.*

Then, we can establish the relationship between θ and t, which is

$\theta = t / 1.41 * 0.52.$

We must bear in mind that it is very important the multiplication by the factor *0.52*, because if not to multiply by this factor, we would not take into account the motion relative to Earth and the encounter will not take place.

Thus, in our editing window GeoGebra, we must re-define θ as

θ = t / 1.41 * 0.52

Now, all the variables are related to the time *t*, so we only have one slider (the slider related to the angle has disappeared from the graphical window).

Now proceed in the graphics window to slide the value of *t*; then, we can observe (in a synchronized "dance") the orbits of all the celestial bodies involved in the simulation, that is, the spacecraft, the Earth and Mars.

We see a graphical window of this "dance" in the figure 6 (this picture is taken when *t = 1.9*).

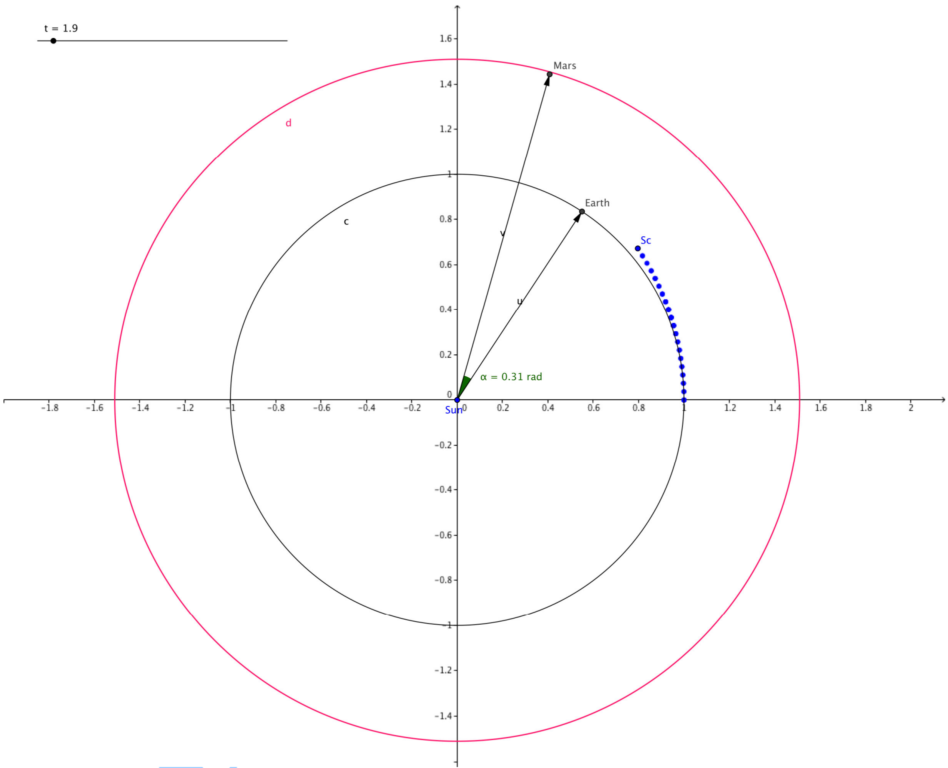

Figure 6. Simulating the journey of the spacecraft to Mars

If we continue sliding the variable *t* until reaching the value *t = 8.5*, we observe the encounter between the spacecraft and the red planet; moreover, observe that the encounter takes place in the apogee position (Fig.7).

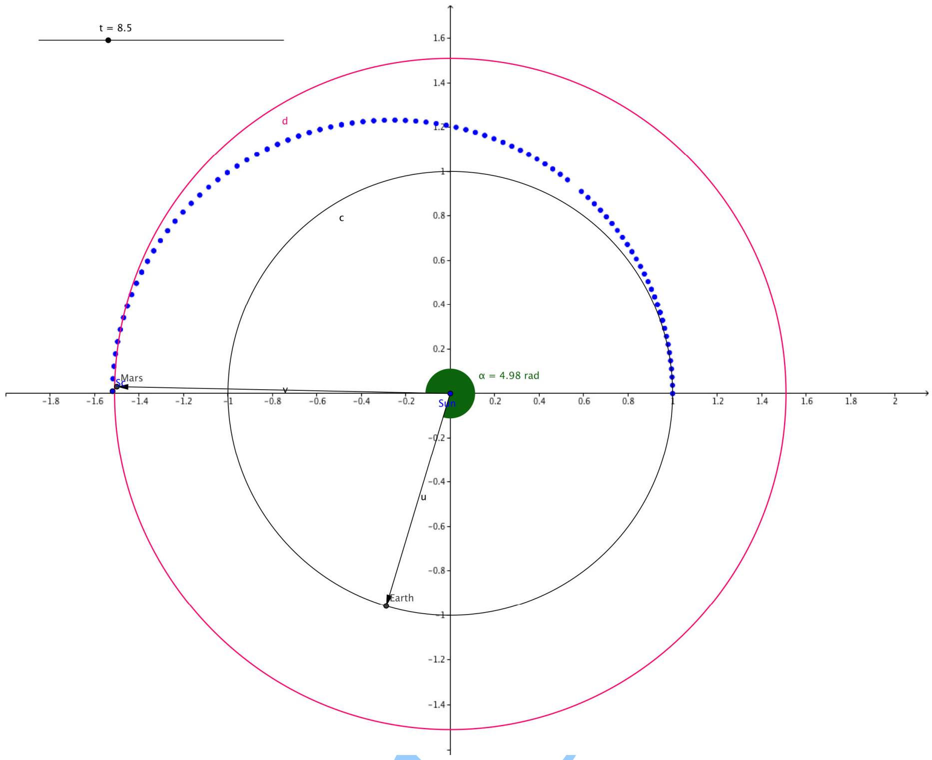

Figure 7. Arriving the spacecraft to Mars 258 days after the launching

Initially, we defined the slider *t* in the range of 0-30. But we want the path of the spacecraft will be no longer visible after the encounter with Mars. How to get that? We only have to introduce a condition in the Object Properties for the slider *t*. Just in the advanced tab, we write $t < 8.5$.

Object Properties → *Advanced* → *Condition to Show Object*

Conclusion and future work

We have computed, under some assumptions, a Hohmann Transfer Orbit to place a spacecraft in the Mar's surface. The period of the journey and the geometric position between the planets have been determined in order to produce an encounter in the orbits. A simulation of the journey has been implemented with GeoGebra, using the excellent characteristics that this software provides us to perform geometric simulations.

In the future we want to complete this simulation calculating the mathematical parameters for the returning journey; that is, we want to determine the trajectory and the launching time to bring back the spacecraft to the earth, following a new Hohmann Transfer Orbit. Besides, we are thinking about the possibility to design other types of orbits to design a trajectory to travel to Mars, using GeoGebra to visualize them.

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